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ANALYSIS OF A NONLOCAL MODEL FOR SPONTANEOUS CELL POLARIZATION*

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Abstract. In this work, we investigate the dynamics of a nonlocal model describing spontaneous cell polarization. It consists of a drift-diffusion equation set in the half-space, with the coupling involving the trace value on the boundary. We characterize the following behaviors in the one-dimensional case: solutions are global if the mass is below the critical mass and they blow up in finite time above the critical mass. The higher-dimensional case is also discussed. The results are reminiscent of the classical Keller–Segel system, but critical spaces are different (L^N instead of $L^{N/2}$ due to the coupling on the boundary). In addition, in the one-dimensional case we prove quantitative convergence results using relative entropy techniques. This work is complemented with a more realistic model that takes into account dynamical exchange of molecular content at the boundary. In the one-dimensional case we prove that blow-up is prevented. Furthermore, density converges toward a nontrivial stationary configuration.

Key words. cell polarization, global existence, blow-up, asymptotic convergence, entropy method, Keller–Segel system

AMS subject classifications. 35B60, 35B44, 35Q92, 92C17, 92B05

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1. Introduction.

Cell polarization refers generically to a process that enables a cell to switch from a spherically symmetric shape to a state with a preferred axis. Such a phenomenon is an essential step for many biological processes in eukaryotic cells and is involved, for instance, in cell migration, division, and morphogenesis. While the precise biochemical basis of polarization can vary greatly, in its early stages polarization is always characterized by an inhomogeneous distribution of specific molecular markers, which in turn induces a mechanical deformation of the cell shape. Cell polarization can be driven by an external asymmetric signal as in the example of chemotaxis, where a chemical gradient imposes the direction of migration of cells [2]. Another example is given by budding yeast (Saccharomyces cerevisiae), for which the external signal is a pheromone gradient, which causes the cell to grow an elongation known as a shmoo in the direction of the mating partner [2]. Besides this mechanism of driven polarization, observations show that some cellular systems, such as mating yeast, can also polarize spontaneously in the absence of external gradients [35]. These two distinct polarization processes, driven or spontaneous, are necessary for cells to fulfill different biological functions and do not exclude each other. While the case of driven polarization can be modeled at least phenomenologically by invoking linear response, the dynamical mechanisms involved in spontaneous polarization are not well

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understood.

The molecular pathways involved in cell polarization have been studied experimentally quite extensively over the past decade, and the obtained results suggest that several alternative mechanisms can be at work. A first mechanism relies on so-called historical marks and assumes that each division event leaves a localized scar with specific molecular markers (Bud1 proteins) which are fixed on the cell membrane and can initiate the next polarization events. A second mechanism proposed in the literature is based on a Turing instability and involves the protein Bem1. In this paper we will focus on a third mechanism which is based on the active transport of polarity markers along cytoskeleton filaments, which was proposed in [35]. The cell cytoskeleton is a network of long semiflexible filaments made up of protein subunits [31]. These filaments (mainly actin or microtubules) act as roads along which motor proteins are able to perform a biased ballistic motion and carry various molecules in a process which consumes the chemical energy of adenosine triphosphate (ATP). In the case of yeast polarization, it is observed that the efficiency of formation of polar caps is reduced when actin transport is disrupted and that the polar caps formed are unstable [35, 36, 22], indicating that actin plays a prominent role in the process. It was proposed that this actin-based mechanism relies on a positive feedback loop in the dynamics of polarity markers, which is mediated by the actin cytoskeleton [35]. Indeed, it was shown that the polarity markers (Cdc42 proteins) activate the polymerization of actin filaments when they are adsorbed on the cell membrane, so that the active transport along filaments pointing toward enriched Cdc42 regions is favored in the cell cytoplasm. The positive feedback loop then results from the fact that the markers Cdc42 themselves are actively transported in the cytoplasm and are therefore preferentially directed toward high concentration regions.

From the physical point of view, achieving an inhomogeneous distribution of diffusing molecules without an external asymmetric field as in the case of spontaneous polarization requires either interactions between the molecules (like in aggregations phenomena or reaction-diffusion systems) or a driving force that maintains the system out of equilibrium. In the case of the cell cytoskeleton, it is well known that the hydrolysis of ATP acts as a sustained energy input which drives the system out of equilibrium, and one can therefore hypothesize on general grounds that spontaneous polarization in cells involves nonequilibrium processes. From the mathematical point of view, spontaneous polarization can be described by so-called local excitation and global inhibition (LEGI) models. For instance, the Turing instability is the basis for several theoretical studies in recent years [21, 25, 29]. Some other studies include cytoskeleton proteins as a limiting factor: local excitation arises from active transport toward the boundary, and global inhibition arises from conservation of mass [18, 29, 35].

Following the work of [18], we study in this article a class of dynamical models for spontaneous cell polarization. We opt for a coarse-grained description of the cytoskeleton as an advection field accounting for active transport. We refer the reader to [37, 12] for further description of the dynamics of actin networks. Here we go beyond the linear stability analysis performed in [18] and analyze the long time asymptotics of the models and discuss the regularity of the solutions. The principal goal of our analysis is to identify regimes in which nonhomogeneous stationary states, which will be interpreted as polarized states, emerge. The main ingredients of the models are as follows. The markers, whose density is denoted by $n(t, x)$, are assumed to diffuse in the cytoplasm and to be actively transported along the cytoskeleton. The resulting motion is a biased diffusion with diffusion coefficient set to 1 and advection field

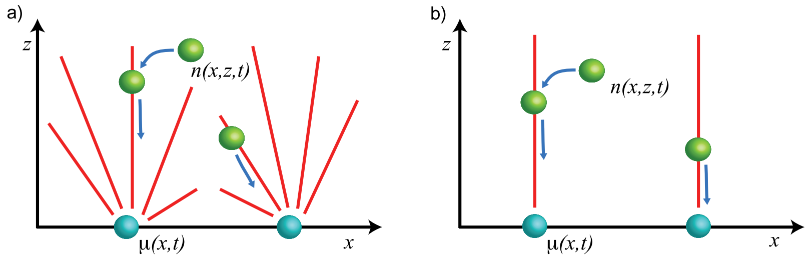

FIG. 1.1. *Schematic representation of the orientation of the cytoskeleton network with respect to the cell membrane. (Left) Actin filaments: Filaments are distributed in several directions from nucleation sites. The resultant field is derived from a harmonic potential with source located on the cell boundary. (Right) Microtubules: Filaments are growing from the center of the cell. The resultant field is normal to the cell boundary.*

$\mathbf{u}(t,x)$. This field is obtained through a coupling with the membrane concentration of markers that is given by the boundary value of $n(t,x)$. In what follows we will consider two representative examples of advection fields. In the first case, called hereafter the *transversal case*, the field $\mathbf{u}(t,x)$ is normal to the domain boundary and schematically models the case of the microtubule cytoskeleton. In the second case, called the *potential case*, the field $\mathbf{u}(t,x)$ derives from a harmonic potential whose source term is located at the boundary. This situation schematically models the case of the actin cytoskeleton which is nucleated from the cell membrane. The two examples are depicted in Figure 1.1. We refer the reader to [18] for a detailed presentation.

The cell is figured by the half-space $\mathcal{H} = \mathbb{R}^{N-1} \times (0,+\infty)$, and we denote the space variable $x = (y,z)$. The time evolution of the molecular markers follows an advection-diffusion equation which reads

$$(1.1)\qquad \begin{cases} \partial_t n(t,x) = \Delta n(t,x) - \nabla \cdot (n(t,x)\mathbf{u}(t,x)) \, , & t>0\, , \quad x \in \mathcal{H}\, , \\ n(0,x) = n_0(x)\, . \end{cases}$$

1.1. The one-dimensional case. We first analyze two different models set on the half-line $(0,+\infty)$. In the simplified version, the advection field is given by $\mathbf{u}(t,z) = -n(t,0)$. Active transport arises at uniform speed, the speed being given by the value of the density at $z = 0$.

1.1.1. The simplified model. The model writes as follows:

$$(1.2)\qquad \partial_t n(t,z) = \partial_{zz} n(t,z) + n(t,0)\partial_z n(t,z)\, , \quad t>0\, , \, z \in (0,+\infty)\, ,$$

together with the zero-flux boundary condition at $z = 0$,

$$(1.3)\qquad \partial_z n(t,0) + n(t,0)^2 = 0.$$

We have formally conservation of molecular content:

$$M = \int_{z>0} n_0(z)\,\mathrm{d}z = \int_{z>0} n(t,z)\,\mathrm{d}z.$$

Solutions of (1.2) may become unbounded in finite time (so-called blow-up). This occurs if the mass M is above the critical mass, $M > 1$. In the case $M < 1$, the solution converges to 0. In the critical case $M = 1$ there exists a family of stationary states parametrized by the first moment. The solution converges to the stationary state corresponding to the first moment of the initial condition.

THEOREM 1.1 (global existence and asymptotic behavior in the subcritical and critical cases: $M \le 1$). *Assume that the initial data n_0 satisfies both $n_0 \in L^1((1+z)\,\mathrm{d}z)$ and $\int_{z>0} n_0(z)(\log n_0(z))_+ \,\mathrm{d}z < +\infty$. Assume in addition that $M \le 1$; then there exists a global solution in the weak sense. The solution satisfies the following estimates for all $T > 0$:*

$$\sup_{t\in(0,T)} \int_{z>0} n(t,z)(\log n(t,z))_+ \,\mathrm{d}z < +\infty,$$
$$\int_0^T \int_{z>0} n(t,z)\left(\partial_z \log n(t,z)\right)^2 \,\mathrm{d}z\,\mathrm{d}t < +\infty.$$

In the subcritical case $M < 1$ the solution strongly converges in L^1 toward the self-similar profile $G_\alpha(y) = \alpha \exp\left(-\alpha y - y^2/2\right)$, where α is uniquely determined by conservation of mass $\int_{y>0} G_\alpha(y)\,dy = M$:

$$\lim_{t\to+\infty} \left\| n(t,z) - \frac{1}{\sqrt{1+2t}} G\left(\frac{z}{\sqrt{1+2t}}\right) \right\|_{L^1} = 0.$$

In the critical case $M = 1$, assuming in addition that the second moment is finite $\int_{z>0} z^2 n_0(z)\,\mathrm{d}z < +\infty$, the solution strongly converges in L^1 toward a stationary state $\alpha\exp(-\alpha z)$, where $\alpha^{-1} = \int_{z>0} z n_0(z)\,\mathrm{d}z$.

THEOREM 1.2 (blow-up of weak solutions: $M > 1$). *Assume $M > 1$. Any weak solution with nonincreasing initial data n_0 blows up in finite time.*

In the present biological context, blow-up of solutions is interpreted as polarization of the cell. Indeed, there is a strong instability driving the system toward an inhomogeneous state. In Figure 1.2 we illustrate Theorems 1.1 and 1.2.

In section 3, we present analogous blow-up results in the case of a finite interval $z \in (0, L)$ or finite range of action (see also Figure 1.3(a)).

Remark 1.3. Such a critical mass phenomenon (global existence *versus* blow-up) has been widely studied for the Keller–Segel system (also known as the Smoluchowski–Poisson system) in two dimensions of space [7, 30]. Equation (1.2) represents in some sense a caricature of the classical Keller–Segel system in the half-line $(0, +\infty)$. Note that there exist other ways to mimic the two-dimensional case in one dimension [11, 13].

Remark 1.4. There is a strong connection between the equation of interest here, (1.2), and the one-dimensional Stefan problem. The latter writes indeed [19] as

$$\begin{cases} \partial_t u(t,z) = \partial_{zz} u(t,z)\,, \quad t > 0\,,\, z \in (-\infty, s(t))\,, \\ \lim_{z\to-\infty} \partial_z u(t,z) = 0\,, \quad u(t,s(t)) = 0\,, \quad \partial_z u(t,s(t)) = -s'(t)\,. \end{cases}$$

The temperature is initially nonnegative: $u(0,z) = u_0(z) \ge 0$. By performing the change of variables $\phi(t,z) = -u(t, s(t) - z)$, we get an equation that is linked to (2.1) by $n(t,z) = \partial_z \phi(t,z)$. This connection provides some insights concerning the possible continuation of solutions after blow-up [19]. This question has raised a lot of interest in recent years [20, 33, 34, 17]. It is postulated in [19] that the one-dimensional Stefan problem is generically noncontinuable after the blow-up time.

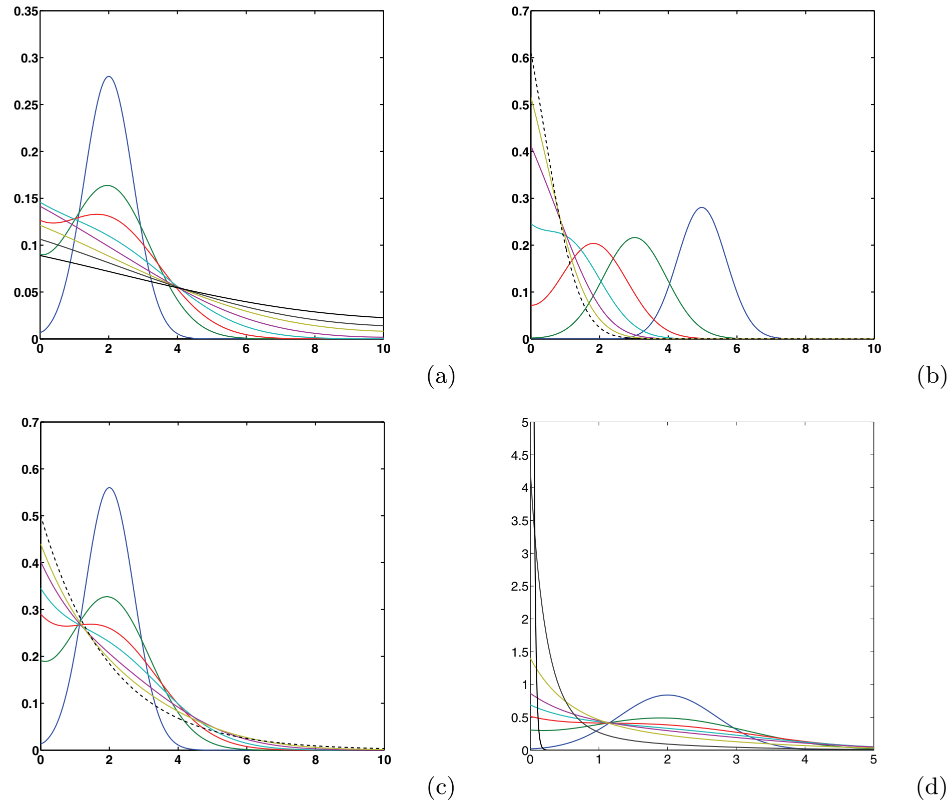

FIG. 1.2. *Asymptotic behavior of solutions to* (1.2)–(1.3) *in the three different regimes.* (a) *Subcritical case: Self-similar decay to zero.* (b) *Subcritical case in the rescaled frame: Convergence toward the self-similar profile G.* (c) *Critical case: Convergence toward the exponential profile with prescribed first momentum.* (d) *Supercritical case: Finite time blow-up. In all cases the initial data is a Gaussian with mass, respectively, $M = 0.5$ (*(a) *and* (b)*), $M = 1$* (c)*, and $M = 1.5$* (d). *In both cases* (b) *and* (c) *the expected stationary state is plotted as the dashed line.*

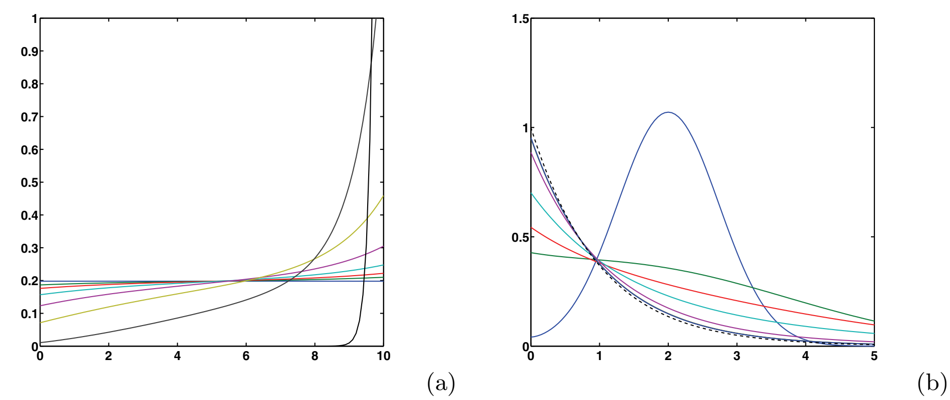

FIG. 1.3. *Asymptotic behavior of the one-dimensional equation with the following configurations:* (a) *A domain with finite size $(0, L)$, where the advection field is given by $\mathbf{u}(t, z) = -n(t, 0) + n(t, L)$ (see section 3.2);* (b) *dynamical exchange of markers at the boundary. In case* (a) *the initial condition is a small perturbation of the constant state. In case* (b) *it is a Gaussian. In both cases the mass is supercritical: $M = 2$.*

1.1.2. The model with dynamical exchange of markers at the boundary. The boundary condition (1.3) turns out to be unrealistic from a biophysical viewpoint. This claim is emphasized by the possible occurrence of blow-up in finite time. On the way toward a more realistic model, we distinguish between cytoplasmic content $n(t,z)$ and the concentration of trapped molecules on the boundary at $z = 0$: $\mu(t)$. Then the exchange of molecules at the boundary is described by very simple kinetics:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu(t) = n(t,0) - \gamma\mu(t)\,.$$

The transport speed is modified accordingly: $\mathbf{u}(t,z) = -\mu(t)$. The model writes as

$$\begin{cases} \partial_t n(t,z) = \partial_{zz} n(t,z) + \mu(t)\partial_z n(t,z)\,, \quad t>0\,,\, z\in(0,+\infty)\,, \\ \partial_z n(t,0) + \mu(t) n(t,0) = \frac{d}{dt}\mu(t)\,. \end{cases}$$

The flux condition on the boundary ensures the conservation of molecular content. Denoting $m(t) = \int_{z>0} n(t,z)\,\mathrm{d}z$ the partial mass of cytoplasmic markers, we have

$$M = \mu_0 + m_0 = \mu(t) + m(t)\,.$$

Since the transport speed is bounded, $\mu(t) \le M$, we clearly have global existence of solutions for any mass $M > 0$. In the following theorem we describe the asymptotic behavior in the supercritical case $M > 1$.

THEOREM 1.5. *Assume that the initial data n_0 satisfies both $n_0 \in L^1((1+z)\,\mathrm{d}z)$ and $\int_{z>0} n_0(z)(\log n_0(z))_+\,\mathrm{d}z < +\infty$. Assume the mass is supercritical, $M > 1$. The partial mass $m(t)$ converges to 1, and the density $n(t,z)$ strongly converges in L^1 toward the exponential profile $(M-1)e^{-(M-1)z}$.*

Theorem 1.5 is illustrated in Figure 1.3(b).

1.2. The higher-dimensional case. In the higher-dimensional case $N \ge 2$ we only partially analyze simplified models such as (1.2), where the transport speed is directly computed from the trace value $n(t,y,0)$. Equation (1.1) is complemented with the zero-flux boundary condition:

$$\partial_z n(t,y,0) - n(t,y,0)\mathbf{u}(t,y,0)\cdot\mathbf{e}_z = 0\,, \quad y \in \mathbb{R}^{N-1}. \tag{1.4}$$

We have formally conservation of the molecular content:

$$M = \int_{\mathcal{H}} n_0(x)\,\mathrm{d}x = \int_{\mathcal{H}} n(t,x)\,\mathrm{d}x.$$

Following [18], we make the distinction between two possible choices for the advection speed $\mathbf{u}$. In the *transversal case*, the field $\mathbf{u}$ is normal to the boundary:

$$\mathbf{u}(t,y,z) = -n(t,y,0)\mathbf{e}_z. \tag{1.5}$$

This corresponds to a particular orientation of the cytoskeleton, modeling the microtubules. Indeed, microtubules are very rigid filaments whose bending length is larger than the typical size of yeast cells [31].

In the *potential case*, the field $\mathbf{u}$ derives from a harmonic potential. The source term of the potential is located on the boundary:

$$\mathbf{u}(t,x) = \nabla c(t,x), \quad \text{where} \quad \begin{cases} -\Delta c(t,x) &= 0, \\ -\partial_z c(t,y,0) &= n(t,y,0). \end{cases} \tag{1.6}$$

This corresponds to another orientation of the cytoskeleton, modeling the actin network. Indeed, the actin network is a diffusive network where orientations are mixed up. In dimension $N = 1$, observe that the two choices (1.5) and (1.6) coincide.

In dimension $N \geq 2$, we state global existence for small initial data. The criteria are identical for the two possible choices of the advection field (1.5) or (1.6). This is a consequence of the two common features: both fields are divergence-free and possess the same normal component at the boundary.

THEOREM 1.6 (global existence in dimension $N \geq 2$). *Assume that the advection field satisfies the following two conditions:* $\nabla \cdot \mathbf{u} \geq 0$ *and* $\mathbf{u}(t, y, 0) \cdot \mathbf{e}_z = n(t, y, 0)$. *Assume that the initial data* n_0 *satisfies both that* $n_0 \in L^1((1 + |x|^2)\,\mathrm{d}x)$ *and that* $\|n_0\|_{L^N}$ *is smaller than some constant* c_N *depending only on the dimension* N. *Then there exists a global weak solution to* (1.1) *and* (1.4).

Notice that both conditions $\nabla \cdot \mathbf{u} \geq 0$ and $\mathbf{u}(t, y, 0) \cdot \mathbf{e}_z = n(t, y, 0)$ are fulfilled in (1.5) and (1.6).

THEOREM 1.7 (blow-up in dimension $N \geq 2$). *Assume that* $n(t, x)$ *is a strong solution to* (1.1) *which verifies the following:*

- $\partial_z n(t, x) \leq 0$ *for all* $x \in \mathcal{H}$ *and* $t > 0$ *when the advective field is given by* (1.5).
- $\partial_z n(t, x) \leq 0$ *and for all* $x \in \mathcal{H}$ *and* $t > 0$*, the matrix* $A(t, x) = x \otimes \partial_z \nabla_y \log n(t, x)$ *satisfies* $A^T + A \geq 0$ *(in the matrix sense) when the advective field is given by* (1.6).

Assume, in addition, that the second momentum is initially small enough: there exists a constant C_N *depending only on the dimension such that* $\int_{x\in\mathcal{H}} |x|^2 n_0(x)\,\mathrm{d}x \leq C_N M^{\frac{N+1}{N-1}}$. *Then the maximal time of existence of the solution is finite.*

Open questions. We end this introductory section with some open questions that we are not able to resolve. (i) Obtain a rate for the convergence in relative entropy in Theorem 1.1 for the cases $M = 1$ and $M < 1$. (ii) Prove blow-up for the systems (1.1)–(1.6) with large initial data without any monotonicity assumption on the density $n(t, x)$ (see also Figure 1.2(d), where blow-up occurs without monotonicity of the initial data).

The outline of the paper is as follows. In section 2, we analyze in full detail the one-dimensional case. In section 3 we study some variants of blow-up criteria in the one-dimensional case. In section 4, we study a model with flux of markers at the boundary in the one-dimensional case. In section 5, we analyze the higher-dimensional case.

Results in the one-dimensional case have been announced in the short note [10].

2. The boundary Keller–Segel (BKS) equation in dimension $N = 1$. In this section we study the equation

$$\begin{cases} \partial_t n(t, z) = \partial_{zz} n(t, z) + n(t, 0)\partial_z n(t, z)\,, & t > 0\,,\ z \in (0, +\infty)\,, \\ \partial_z n(t, 0) + n(t, 0)^2 = 0\,, \end{cases} \tag{2.1}$$

and we prove Theorems 1.1 and 1.2. More precisely, in section 2.1 we prove the existence of a global weak solution for $M \leq 1$. Then in section 2.3 we prove the blow-up character in the case $M > 1$.

We begin with a proper definition of weak solutions, adapted to our context.

DEFINITION 2.1. *We say that* $n(t, z)$ *is a weak solution of* (2.1) *on* $(0, T)$ *if it satisfies*

$$n \in L^\infty(0, T; L^1_+(\mathbb{R}_+))\,, \quad \partial_z n \in L^1((0, T) \times \mathbb{R}_+)\,, \tag{2.2}$$

and $n(t,z)$ is a solution of (2.1) *in the sense of distributions in $\mathcal{D}'(\mathbb{R}_+)$.*

Since the flux $(\partial_z n(t,z)+n(t,0)n(t,z))$ belongs to $L^1((0,T)\times\mathbb{R}_+)$, the solution is well defined in the distributional sense under assumption (2.2). In fact, we can write $\int_0^T n(t,0)\,\mathrm{d}t = -\int_0^T\int_{z>0}\partial_z n(t,z)\,\mathrm{d}z\,\mathrm{d}t$.

Weak solutions in the sense of Definition 2.1 are mass-preserving:

$$M = \int_{z>0} n_0(z)\,\mathrm{d}z = \int_{z>0} n(t,z)\,\mathrm{d}z.$$

The proof closely follows the arguments of the next lemma, which is concerned with moment growth.

LEMMA 2.2 (moment growth). *Assume $n(t,z)$ is a weak solution of* (2.1). *Assume in addition that $zn_0 \in L^1(\mathbb{R}_+)$. Then the following identity holds true:*

$$\int_{z>0} zn(T,z)\,\mathrm{d}z = \int_{z>0} zn_0(z)\,\mathrm{d}z + \int_0^T \left(1 - \int_{z>0} n(t,z)\,\mathrm{d}z\right) n(t,0)\,\mathrm{d}t. \tag{2.3}$$

Proof. Consider the approximation function $\chi(z)$ which verifies $\chi(z)=1$ if $0\le z\le 1$, $\chi(z)=0$ if $z\ge 2$, which is smooth and nonnegative everywhere. Define the family of functions $(\varphi_\varepsilon)_\varepsilon$ by $\varphi_\varepsilon(z) = z\chi(\varepsilon z)$. We recall the weak formulation

$$\begin{aligned}\int_{z>0} n(T,z)\varphi_\varepsilon(z)\,\mathrm{d}z = &\int_{z>0} n_0(z)\varphi_\varepsilon(z)\,\mathrm{d}z\\ &- \int_0^T\int_{z>0}\left(\partial_z n(t,z) + n(t,0)n(t,z)\right)\varphi_\varepsilon'(z)\,\mathrm{d}z\,\mathrm{d}t.\end{aligned}$$

The function $\varphi_\varepsilon(z)$ converges monotonically to z as $\varepsilon\to 0$; hence, from the monotone convergence theorem, we deduce that $zn(T,z)\in L^1$.

The function $\varphi_\varepsilon'(z) = \chi(\varepsilon z) + \varepsilon z\chi'(\varepsilon z)$ is bounded in L^∞ uniformly in ε, and it converges to 1 a.e. Since $n(\cdot,0)n \in L^1((0,T)\times R_+)$ and $\partial_z n \in L^1((0,T)\times\mathbb{R}_+)$, from Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^T\int_{z>0}\varphi_\varepsilon'(z)n(t,0)n(t,z)\,\mathrm{d}z\,\mathrm{d}t &= \int_0^T\int_{z>0} n(t,0)n(t,z)\,\mathrm{d}z\,\mathrm{d}t,\\ \lim_{\varepsilon\to 0}\int_0^T\int_{z>0}\varphi_\varepsilon'(z)\,\partial_z n(t,z)\,\mathrm{d}z\,\mathrm{d}t &= \int_0^T\int_{z>0}\partial_z n(t,z)\,\mathrm{d}z\,\mathrm{d}t = -\int_0^T n(t,0)\,\mathrm{d}t.\end{aligned}$$ ∎

2.1. Global existence for subcritical mass $M < 1$.

2.1.1. A priori estimates. Our next result is concerned with the derivation of a priori bounds for solutions to (2.1) in the classical sense.

PROPOSITION 2.3 (main a priori estimate). *Let n be a classical solution to* (2.1). *If $M<1$, then the following estimate holds true for some $\delta>0$ and for all $t\in(0,T)$:*

$$\begin{aligned}&\int_{z>0} n(t,z)(\log n(t,z))_+\,\mathrm{d}z + \delta\int_0^t\int_{z>0} n(s,z)(\partial_z\log n(s,z))^2\,\mathrm{d}z\,\mathrm{d}s\\ &\qquad\le \int_{z>0} n_0(z)\left(\log n_0(z)\right)_+\,\mathrm{d}z + \int_{z>0} zn_0(z)\,\mathrm{d}z + C(T)\,.\end{aligned} \tag{2.4}$$

Proof. We first derive the following trace-type inequality:

$$\begin{aligned}n(t,0)^2 &= \left(\int_{z>0}\partial_z n(t,z)\,\mathrm{d}z\right)^2\\ &\le \left(\int_{z>0} n(t,z)\,\mathrm{d}z\right)\left(\int_{z>0} n(t,z)\left(\partial_z\log n(t,z)\right)^2\,\mathrm{d}z\right).\end{aligned} \tag{2.5}$$

We compute the evolution of the entropy:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{z>0} n(t,z)\log n(t,z)\,\mathrm{d}z &= \int_{z>0} \partial_t n(t,z)\log n(t,z)\,\mathrm{d}z\\ &= -\int_{z>0} \left(\partial_z n(t,z) + n(t,0)n(t,z)\right)\frac{\partial_z n(t,z)}{n(t,z)}\,\mathrm{d}z\\ &= -\int_{z>0} n(t,z)\left(\partial_z \log n(t,z)\right)^2\,\mathrm{d}z + n(t,0)^2\,. \end{aligned} \tag{2.6}$$

The two contributions are competing. We estimate the balance using inequality (2.5):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{z>0} n(t,z)\log n(t,z)\,\mathrm{d}z \le (M-1)\int_{z>0} n(t,z)\left(\partial_z \log n(t,z)\right)^2\,\mathrm{d}z.$$

On the contrary to the classical two-dimensional Keller–Segel equation, the dissipation of entropy gives directly the sharp criterion on the mass. There is no need to seek a free energy as in [7] (and references therein). To control the negative part of the entropy, we use the following lemma adapted from [7, 9].

LEMMA 2.4. *For any $f \in L^1_+(\mathbb{R}_+, (1+z)\,\mathrm{d}z)$, if $\int f\log f < +\infty$, then $f\log f$ is in $L^1(\mathbb{R}_+)$, and for all $\alpha > 0$, the following inequality holds true:*

$$\int_{z>0} f(z)(\log f(z))_+\,\mathrm{d}z \le \int_{z>0} f(z)\left(\log f(z) + \alpha z\right)\,\mathrm{d}z + \frac{1}{\alpha e}\,. \tag{2.7}$$

Proof. Let $\overline{f} = f\mathbf{1}_{f\le 1}$ and $m = \int_{z>0}\overline{f}(z)\,\mathrm{d}z$. We build up the relative entropy between $\overline{f}$ and $\alpha e^{-\alpha z}$.

$$\int_{z>0}\overline{f}(z)\left(\log\overline{f}(z) + \alpha z\right)\,\mathrm{d}z = \int_{z>0}\frac{\overline{f}(z)}{\alpha e^{-\alpha z}}\log\left(\frac{\overline{f}(z)}{\alpha e^{-\alpha z}}\right)\alpha e^{-\alpha z}\,\mathrm{d}z + m\log\alpha.$$

Using Jensen's inequality, we deduce that

$$\begin{aligned}&\int_{z>0}\frac{\overline{f}(z)}{\alpha e^{-\alpha z}}\log\left(\frac{\overline{f}(z)}{\alpha e^{-\alpha z}}\right)\alpha e^{-\alpha z}\,\mathrm{d}z\\ &\qquad\ge \left(\int_{z>0}\frac{\overline{f}(z)}{\alpha e^{-\alpha z}}\alpha e^{-\alpha z}\,\mathrm{d}z\right)\log\left(\int_{z>0}\frac{\overline{f}(z)}{\alpha e^{-\alpha z}}\alpha e^{-\alpha z}\,\mathrm{d}z\right)\\ &\qquad= m\log m.\end{aligned}$$

Therefore,

$$\int_{z>0}\overline{f}(z)\log\overline{f}(z)\,\mathrm{d}z + \alpha\int_{z>0} z\overline{f}(z)\,\mathrm{d}z \ge m\log(\alpha m) \ge -\frac{1}{\alpha e}\,.$$

Using

$$\int_{z>0} f(z)(\log f(z))_+\,\mathrm{d}z = \int_{z>0} f(z)\log f(z)\,\mathrm{d}z - \int_{z>0}\overline{f}(z)\log\overline{f}(z)\,\mathrm{d}z,$$

this completes the proof of Lemma 2.4. ◻

Let us now estimate the first moment. Recalling (2.3), we deduce that

$$\begin{aligned}\int_{z>0} zn(t,z)\,\mathrm{d}z &\le \int_{z>0} zn_0(z)\,\mathrm{d}z + \int_0^t n(s,0)\,\mathrm{d}s \\ &\le \int_{z>0} zn_0(z)\,\mathrm{d}z + \frac{T}{4\delta'} + \delta' \int_0^t n(s,0)^2\,\mathrm{d}s \\ &\le \int_{z>0} zn_0(z)\,\mathrm{d}z + \frac{T}{4\delta'} \\ &\qquad + \delta' \int_0^t \int_{z>0} n(s,z)\left(\partial_z \log n(s,z)\right)^2\,\mathrm{d}z\,\mathrm{d}s. \end{aligned} \tag{2.8}$$

Combining (2.6), (2.7), and (2.8) with $\alpha = 1$, we obtain that

$$\begin{aligned}\int_{z>0} n(t,z)\left(\log n(t,z)\right)_+\,\mathrm{d}z + (1-M-\delta')\int_0^t \int_{z>0} n(s,z)\left(\partial_z \log n(s,z)\right)^2\,\mathrm{d}z\,\mathrm{d}s \\ \le \int_{z>0} n_0(z)\log n_0(z)\,\mathrm{d}z + \int_{z>0} zn_0(z)\,\mathrm{d}z + \frac{1}{e} + \frac{T}{4\delta'}\,.\end{aligned}$$

Since $M < 1$, we can choose $\delta' > 0$ such that (2.4) holds. ∎

2.1.2. Regularization procedure. To prove existence of weak solutions in the sense of Definition 2.1, we perform a classical regularization procedure. We carefully choose our function spaces in order to end up with minimal assumptions on the initial data. We introduce

$$a^\varepsilon(t) = \int_{z>0} \phi_\varepsilon(z) n^\varepsilon(t,z)\,\mathrm{d}z,$$

where ϕ_ε is an approximation to the identity. We have formally $a^\varepsilon(t) \to n(t,0)$ as $\varepsilon \to 0$.

We consider the following regularized problem:

$$\begin{cases} \partial_t n^\varepsilon(t,z) = \partial_{zz} n^\varepsilon(t,z) + a^\varepsilon(t)\partial_z n^\varepsilon(t,z)\,, \\ \partial_z n^\varepsilon(t,0) + a^\varepsilon(t) n^\varepsilon(t,0) = 0. \end{cases} \tag{2.9}$$

Our aim is to extend the main a priori estimate (2.4) to the regularized problem (2.9). We check that

$$a^\varepsilon(t) = -\int_{z>0} \phi_\varepsilon(z) \int_{y=z}^{+\infty} \partial_z n^\varepsilon(t,y)\,\mathrm{d}y\,\mathrm{d}z \le \int_{z>0} \left|\partial_z n^\varepsilon(t,z)\right|\mathrm{d}z.$$

Thus the following inequality replaces (2.5):

$$a^\varepsilon(t) n^\varepsilon(t,0) \le M\left(\int_{z>0} n^\varepsilon(t,z)\left(\partial_z \log n^\varepsilon(t,z)\right)^2\,\mathrm{d}z\right).$$

On the other hand, the moment growth estimate relies only on the diffusion contribution. We have accordingly

$$\begin{aligned}\int_{z>0} zn^\varepsilon(t,z)\,\mathrm{d}z \le \int_{z>0} zn_0(z)\,\mathrm{d}z + \frac{T}{4\delta'} \\ + \delta' \int_0^t \int_{z>0} n^\varepsilon(s,z)\left(\partial_z \log n^\varepsilon(s,z)\right)^2\,\mathrm{d}z\,\mathrm{d}s.\end{aligned} \tag{2.10}$$

It is then straightforward to justify (2.4) for the regularized solution n^ε along the lines of Proposition 2.3. There exists $\delta > 0$ such that

$$\begin{aligned} &\int_{z>0} n^\varepsilon(t,z)(\log n^\varepsilon(t,z))_+ \,\mathrm{d}z + \delta \int_0^t \int_{z>0} n^\varepsilon(s,z)(\partial_z \log n^\varepsilon(s,z))^2 \,\mathrm{d}s\,\mathrm{d}z \\ &\qquad \le \int_{z>0} n_0(z)\,(\log n_0(z))_+ \,\mathrm{d}z + \int_{z>0} z n_0(z)\,\mathrm{d}z + C(T)\,. \end{aligned} \tag{2.11}$$

2.1.3. Time compactness. Passing to the limit as $\varepsilon \to 0$, the main difficulty lies in the nonlinear term $a^\varepsilon(t)\partial_z n^\varepsilon(t,z)$. We need some compactness to proceed further. It is provided by the Aubin–Simon lemma; see [3, 26, 32].

LEMMA 2.5 (Aubin–Simon). *Let $X \subset B \subset Y$ be Banach spaces such that the embedding $X \subset B$ is compact. Assume that the set of functions $\mathcal{F}$ satisfies the following: $\mathcal{F}$ is bounded in $L^2(0,T;X)$ and $\partial_t f$ is uniformly bounded in $L^2(0,T;Y)$. Then $\mathcal{F}$ is relatively compact in $L^2(0,T;B)$.*

The natural choice for spaces in our context would be $\widetilde{X} = W^{1,1}(\mathbb{R}_+)$ and $B = \mathcal{C}^0(\mathbb{R}_+)$ (up to the decay problem at infinity). However, due to the possible apparition of jumps, the embedding $\tilde{X} \subset B$ is not compact. Using the entropy estimate (2.11), we are able to modify the space $\widetilde{X}$ in order to make the embedding $X \subset B$ compact. The crucial point is to obtain an equicontinuity condition weaker than any Hölder condition, in the spirit of [1, Theorem 8.36].

LEMMA 2.6. *Assume $\mathcal{F}$ is a set of nonnegative bounded functions in the following sense: there exists a constant $A > 0$ such that for all $f \in \mathcal{F}$*

$$\sup_{t\in(0,T)} \int_{z>0} f(t,z)(\log f(t,z))_+ \,\mathrm{d}z \le A\,, \qquad \int_0^T \int_{z>0} f(t,z)\,(\partial_z \log f(t,z))^2 \,\mathrm{d}z\,\mathrm{d}t \le A.$$

Then there exist a continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant A' depending on A such that $\eta(0) = 0$ and for all functions $f \in \mathcal{F}$ we have

$$\int_0^T \left(\sup_{x\neq y} \frac{|f(t,y) - f(t,x)|}{\eta(y-x)} \right)^2 \mathrm{d}t \le A'. \tag{2.12}$$

Proof. First, for $x < y$ we have that

$$\begin{aligned} |f(t,y) - f(t,x)|^2 &\le \left(\int_x^y |\partial_z f(t,z)|\,\mathrm{d}z \right)^2 \\ &\le \left(\int_x^y f(t,z)\,\mathrm{d}z \right) \left(\int_{z>0} f(t,z)\,(\partial_z \log f(t,z))^2 \,\mathrm{d}z \right). \end{aligned} \tag{2.13}$$

We use the Jensen inequality for $x < y$:

$$\begin{aligned} \left(\frac{1}{y-x} \int_x^y f(t,z)\,\mathrm{d}z \right) \log \left(\frac{1}{y-x} \int_x^y f(t,z)\,\mathrm{d}z \right)_+ &\le \frac{1}{y-x} \int_x^y f(t,z)(\log f(t,z))_+ \,\mathrm{d}z \\ &\le \frac{A}{y-x}\,. \end{aligned}$$

We can invert this inequality to get

$$\frac{1}{y-x} \int_x^y f(t,z)\,\mathrm{d}z \le \Phi\left(\frac{A}{y-x} \right), \tag{2.14}$$

where $\Phi : [0,+\infty) \to [1,+\infty)$ is the reciprocal bijection of $x(\log x)_+$. We define $\eta(z) = z\Phi(z^{-1}A)$. Clearly $\eta(z) \to 0$ as $z \to 0$ since Φ is sublinear. Combining (2.13) and (2.14), we deduce the estimate (2.12). $\square$

We denote $\mathcal{C}^{0,\eta}$ as the space of functions having modulus of continuity controlled by η:

$$\mathcal{C}^{0,\eta} = \left\{ g \in \mathcal{C}^0 \ : \sup_{x \neq y} \frac{|g(y) - g(x)|}{\eta(y-x)} < +\infty \right\}.$$

The injection $\mathcal{C}^{0,\eta} \subset \mathcal{C}^0$ is compact on bounded intervals [1]. The behavior of functions outside bounded intervals in our context is controlled by the following estimate, which is a consequence of (2.13) as $y \to +\infty$:

$$|f(t,x)|^2 \leq \frac{1}{x} \left(\int_{z>0} z f(t,z)\,\mathrm{d}z \right) \left(\int_{z>0} f(t,z) \left(\partial_z \log f(t,z)\right)^2 \,\mathrm{d}z \right). \tag{2.15}$$

The last requirement in the Aubin–Simon lemma consists in getting a very weak estimate for the time derivative $\partial_t n^\varepsilon$. We can write

$$\partial_t n^\varepsilon(t,z) + \partial_z j^\varepsilon(t,z) = 0,$$

where $j^\varepsilon(t,z) = \partial_z n^\varepsilon(t,z) + a^\varepsilon(t) n^\varepsilon(t,z)$ is uniformly bounded in $L^2(0,T;L^1(\mathbb{R}_+))$, due to (2.11) and the following inequalities:

$$\|a^\varepsilon\|^2_{L^2(0,T)} \leq \|\partial_z n^\varepsilon\|^2_{L^2(0,T;L^1(\mathbb{R}_+))} \leq M \int_0^T \int_{z>0} n^\varepsilon(t,z)(\partial_z \log n^\varepsilon(t,z))^2 \,\mathrm{d}t\,\mathrm{d}z.$$

Hence $\partial_t n^\varepsilon$ is uniformly bounded in $L^2\left(0,T;(W^{1,\infty}(\mathbb{R}_+))'\right)$.

We introduce some useful functional spaces, endowed with their corresponding norms:

$$\begin{aligned} X &= \{g \in \mathcal{C}^{0,\eta}(\mathbb{R}_+) : \ z^{1/2} g(z) \in L^\infty(\mathbb{R}_+)\}, \\ B &= \mathcal{C}^0(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+), \\ Y &= \left(W^{1,\infty}(\mathbb{R}_+)\right)'. \end{aligned}$$

It is straightforward to check that X is compactly embedded in B.

Combining the above estimates, (2.10)–(2.11)–(2.12)–(2.15), we obtain that n^ε is bounded in $L^2(0,T;X)$ uniformly with respect to ε. The Aubin–Simon lemma ensures that, up to extracting a subsequence, n^ε converges strongly in $L^2(0,T;B)$ toward some n. From uniform convergence of n^ε, we deduce that $a^\varepsilon(t) \to n(t,0)$ strongly in $L^2(0,T)$. Hence we can pass to the limit in the nonlinear term $a^\varepsilon(t) n^\varepsilon(t,z)$ in the weak formulation.

To conclude we verify that the a priori estimates given in Proposition 2.3 are valid after passing to the limit $\varepsilon \to 0$. From the strong convergence in $L^2(0,T;B)$ we deduce that, up to extracting a subsequence that we do not relabel,

$$\lim_{\varepsilon \to 0} \int_{z>0} n^\varepsilon(t,z) \left(\log n^\varepsilon(t,z)\right)_+ \,\mathrm{d}z = \int_{z>0} n(t,z) \left(\log n(t,z)\right)_+ \,\mathrm{d}z, \quad \text{a.e. } t \in (0,T)\,.$$

On the other hand, we use the convex character of the functional (see [7] and the references therein) $\int_{z>0} f(z) \left(\partial_z \log f(z)\right)^2 \,\mathrm{d}z = 4 \int_{z>0} \left(\partial_z \sqrt{f(z)}\right)^2 \,\mathrm{d}z$. We have, finally,

$$\liminf_{\varepsilon \to 0} \int_0^t \int_{z>0} n^\varepsilon(s,z) \left(\partial_z \log n^\varepsilon(s,z)\right)^2 \,\mathrm{d}z\,\mathrm{d}s \geq \int_0^t \int_{z>0} n(s,z) \left(\partial_z \log n(s,z)\right)^2 \,\mathrm{d}z\,\mathrm{d}s.$$

So the a priori estimate (2.4) is valid a.e. $t \in (0, T)$.

2.2. Long-time behavior for the critical and the subcritical cases. In this section, we investigate the long-time behavior of solutions in the case $M \leq 1$ using entropy methods. We distinguish between the critical case (no need to rescale) and the subcritical case (self-similar diffusive scaling).

2.2.1. The critical case: Global existence and asymptotic convergence. The main inequality we have used so far in order to prove global existence is (2.5). Equality occurs if $\log n(t,z)$ is linear w.r.t. z: there exists $\alpha(t) > 0$ such that $n(t,z) = M\alpha(t)\exp(-\alpha(t)z)$. In fact, the boundary condition (2.1) implies $M = 1$. On the other hand, the stationary states to (2.1) are precisely the one-parameter family:

$$h_\alpha(z) = \alpha \exp\left(-\alpha z\right), \quad \alpha > 0.$$

This motivates us to introduce the relative entropy:

$$\mathbf{H}(t) = \int_{z>0} \frac{n(t,z)}{h_\alpha(z)} \log\left(\frac{n(t,z)}{h_\alpha(z)}\right) h_\alpha(z)\,\mathrm{d}z = \int_{z>0} n(t,z)\log n(t,z)\,\mathrm{d}z + \alpha\mathbf{J}(t) - \log\alpha.$$

Recalling (2.3), we notice that the first momentum of density is conserved in the case $M = 1$: $\mathbf{J}(t) = \mathbf{J}(0)$. This prescribes the value for α provided we can pass to the limit $t \to \infty$: $\alpha^{-1} = \mathbf{J}(0)$. We also recall the formal computation giving the time evolution of the relative entropy (2.6):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}(t) &= -\int_{z>0} n(t,z)\left(\partial_z \log n(t,z)\right)^2\,\mathrm{d}z + n(t,0)^2 \\ &= -\int_{z>0} n(t,z)\left(\partial_z \log n(t,z) + n(t,0)\right)^2\,\mathrm{d}z \leq 0. \end{aligned} \tag{2.16}$$

The Jensen inequality yields $\mathbf{H}(t) \geq 0$, so we have $0 \leq \mathbf{H}(t) \leq \mathbf{H}(0)$. We deduce from Lemma 2.4 that the quantity $\int_{z>0} n(t,z)(\log n(t,z))_+\,\mathrm{d}z$ is uniformly bounded by some constant denoted by C_0:

$$\int_{z>0} n(t,z)(\log n(t,z))_+\,\mathrm{d}z \leq C_0, \quad \text{a.e. } t \in (0,+\infty)\,.$$

The method of proving convergence in relative entropy toward h_α is as follows. We first gain a priori estimates which enable passing to the limit after extraction as in section 2.1.3. For this we update the estimates in section 2.1 with the key information that the entropy $\mathbf{H}$ is uniformly bounded. The identification of the limit requires more information concerning the behavior of the density at infinity. We use the fact that the first momentum drives the evolution of the second one. Finally, we conclude that the entropy converges to 0 along some subsequence. Since it is nonincreasing, it converges to 0 globally.

A priori bound. We cannot follow the strategy developed in section 2.1 since we crucially used $M < 1$. We need to gain some control on the dissipation (2.16) which is the competition of two opposite contributions that are nearly equally balanced. For that purpose we introduce the function $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Lambda(0) = 0$ and $\Lambda'(u) = (\log u)_+^{1/2}$. It is nondecreasing, convex, and superlinear. Thus there exists

$A \in R$ such that $\Lambda(u)^2 \geq 2C_0u^2$ for all $u \geq A$. Adapting (2.5) to our context, we get

$$\begin{aligned}
\Lambda(n(t,0))^2 &= \left(-\int_{z>0} \partial_z \left(\Lambda(n(t,z))\right) \mathrm{d}z\right)^2 \\
&= \left(-\int_{z>0} \Lambda'(n(t,z))n(t,z)\partial_z(\log n(t,z)) \, \mathrm{d}z\right)^2 \\
&\leq \left(\int_{z>0} n(t,z)|\Lambda'(n(t,z))|^2 \, \mathrm{d}z\right) \left(\int_{z>0} n(t,z) \left(\partial_z \log n(t,z)\right)^2 \, \mathrm{d}z\right) \\
&\leq \left(\int_{z>0} n(t,z)(\log n(t,z))_+ \, \mathrm{d}z\right) \left(\int_{z>0} n(t,z) \left(\partial_z \log n(t,z)\right)^2 \, \mathrm{d}z\right) \\
&\leq C_0 \int_{z>0} n(t,z) \left(\partial_z \log n(t,z)\right)^2 \, \mathrm{d}z. \qquad (2.17)
\end{aligned}$$

From (2.16) and (2.17), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{z>0} n(t,z) \log n(t,z) \, \mathrm{d}z \leq \begin{cases} 0 & \text{if} \quad n(t,0) \leq A, \\ -\frac{\Lambda(n(t,0))^2}{C_0} + n(t,0)^2 \leq -n(t,0)^2 & \text{if} \quad n(t,0) \geq A. \end{cases}$$

We introduce the set $E = \{t : n(t,0) \geq A\}$. We have obtained the estimate

$$\int_E n(t,0)^2 \, \mathrm{d}t \leq \int_{z>0} n_0(z) \log n_0(z) \, \mathrm{d}z; \qquad (2.18)$$

thus $n(t,0)$ cannot be too large (in the L^2 sense).

We deduce from (2.16) and (2.18) that $\int_0^t \int_{z>0} n(s,z) \left(\partial_z \log n(s,z)\right)^2 \, \mathrm{d}z \, \mathrm{d}s$ is bounded for all $t \in (0,T)$. The previous statements prove that Proposition 2.3 remains valid in the case $M = 1$. Next, the existence proof, in the case $M = 1$, is similar to the case $M < 1$, and we do not repeat it here.

Passing to the limit. Let N be any integer. We translate the solution in time: we define $u_N(s,x) = n(N+s,x)$. The function $\mathbf{H}(t)$ is nonincreasing and bounded below by zero. Therefore, the entropy dissipation (2.16) converges to zero in an averaged sense. The estimate

$$\int_N^{N+1} \left(\int_{z>0} n(t,z) \left(\partial_z \log n(t,z)\right)^2 \, \mathrm{d}z - n(t,0)^2\right) \mathrm{d}t = \mathbf{H}(N) - \mathbf{H}(N+1) \xrightarrow[N\to\infty]{} 0$$

reads

$$\int_0^1 \left(\int_{z>0} u_N(s,z) \left(\partial_z \log u_N(s,z)\right)^2 \, \mathrm{d}z - u_N(s,0)^2\right) \mathrm{d}s \xrightarrow[N\to\infty]{} 0. \qquad (2.19)$$

We deduce from (2.18) that $u_N(s,0)$ is bounded in $L^2(0,1)$ uniformly w.r.t. N. Hence both terms are bounded in (2.19). This enables passing to the limit as in section 2.1.3. Up to extracting a subsequence (labeled with N') there exists u_∞ such that $u_{N'} \to u_\infty$ strongly in $L^2(0,1;B)$:

$$\int_0^1 \|u_{N'}(s) - u_\infty(s)\|_B^2 \, \mathrm{d}s \to 0. \qquad (2.20)$$

We can pass to the limit in each term of the averaged dissipation:

$$0 = \liminf_{N'\to\infty} \int_0^1 \left(\int_{z>0} u_{N'}(s,z) \left(\partial_z \log u_{N'}(s,z)\right)^2 \, \mathrm{d}z - u_{N'}(s,0)^2 \right) \mathrm{d}s$$
$$\geq \int_0^1 \left(\int_{z>0} u_{\infty}(s,z) \left(\partial_z \log u_{\infty}(s,z)\right)^2 \, \mathrm{d}z - u_{\infty}(s,0)^2 \right) \mathrm{d}s \geq 0.$$

We have used the $L^2(0,T;L^\infty(\mathbb{R}_+))$ strong convergence (see expression (2.20)) to pass to the limit in the nonlinear term $u_{N'}(s,0)^2$, and also the convexity of the functional $\int_{z>0} f(z) \left(\partial_z \log f(z)\right)^2 \, \mathrm{d}z$ (see section 2.1.3).

Identification of the limit. We deduce that u_∞ satisfies a.e.

$$u_\infty(s,z) = \beta(s)\exp(-\alpha(s)z)\,, \quad \alpha(s), \beta(s) > 0\,.$$

To determine $\alpha(s)$ and $\beta(s)$ we shall use the conservations of mass and first momentum. Since the first momentum is uniformly bounded, we have that $M = \lim \int_{z>0} u_{N'}(t,z)\,\mathrm{d}z = \int_{z>0} u_\infty(t,z)\,\mathrm{d}z$. This yields $\alpha(s) = \beta(s)$.

We have proved so far that we can always extract a subsequence such that $u_{N'}(s,z)$ approaches $u_\infty(s,z)$ in $L^2(0,T;B)$. We explain below why it is delicate to derive $\alpha(s) = \alpha = \mathbf{J}(0)^{-1}$ without any better control of the density $n(t,z)$ as $z \to +\infty$. Suppose we have $\alpha(s) \equiv \overline{\alpha}$, and the convergence $u_N(s,z) \to u_\infty(z)$ is uniform. We would have, on the one hand,

$$\alpha^{-1} = \liminf \int_{z>0} z u_{N'}(t,z)\,\mathrm{d}z \geq \int_{z>0} z u_\infty(z)\,\mathrm{d}z = (\overline{\alpha})^{-1} \tag{2.21}$$

and, on the other hand,

$$0 \leq \lim \mathbf{H}(t) = \int_{z>0} u_\infty(z)\log u_\infty(z)\,\mathrm{d}z + 1 - \log\alpha = \log\overline{\alpha} - \log\alpha.$$

We would deduce $\overline{\alpha} \geq \alpha$, which is the same as (2.21).

In the case $\int_{z>0} z^2 n_0(z)\,\mathrm{d}z < +\infty$, let us examine the evolution of the second momentum. We simply have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{z>0} z^2 n(t,z)\,\mathrm{d}z = M - n(t,0)\mathbf{J}(t) = 1 - n(t,0)\alpha^{-1}\,. \tag{2.22}$$

The idea is to pass to the pointwise limit $n(t,0) \to \overline{\alpha}$. If $\overline{\alpha} > \alpha$, the right-hand side of (2.22) becomes asymptotically $1 - \overline{\alpha}\alpha^{-1} < 0$, which leads to a contradiction.

Let us introduce the notation $\mathbf{I}(t) = \int_{z>0}(z^2/2)n(t,z)\,\mathrm{d}z$. We have

$$\mathbf{I}(N+1) - \mathbf{I}(N) = \int_N^{N+1} \left(1 - n(t,0)\alpha^{-1}\right)\,\mathrm{d}t = \int_0^1 \left(1 - u_N(s,0)\alpha^{-1}\right)\,\mathrm{d}s.$$

Since $\mathbf{I}$ is a nonnegative quantity, we clearly have $\limsup \mathbf{I}(N+1) - \mathbf{I}(N) \geq 0$. Furthermore, we have $\limsup \mathbf{I}(N+1) - \mathbf{I}(N) \leq 0$. To see this, assume on the contrary that $\limsup \mathbf{I}(N+1) - \mathbf{I}(N) = \delta > 0$. We can extract a converging subsequence. Keeping the same notation as above, we have $\lim \mathbf{I}(N'+1) - \mathbf{I}(N') = \delta$. We can pass to the limit similarly (up to further extracting) in the following average quantities:

$$\alpha^{-1} = \liminf \int_0^1 \int_{z>0} z u_{N'}(t,z)\,\mathrm{d}z\,\mathrm{d}s$$

$$\geq \int_0^1 \int_{z>0} z u_\infty(s,z)\,\mathrm{d}z\,\mathrm{d}s = \int_0^1 (\alpha(s))^{-1}\,\mathrm{d}s, \tag{2.23}$$

$$\delta = \lim \int_0^1 \left(1 - u_{N'}(s,0)\alpha^{-1}\right)\,\mathrm{d}s = 1 - \alpha^{-1}\int_0^1 \alpha(s)\,\mathrm{d}s. \tag{2.24}$$

Inequality (2.23) yields $\int_0^1 \alpha(s)\,\mathrm{d}s \geq \alpha$ by Jensen's inequality. This is in contradiction to (2.24).

We conclude that $\limsup \mathbf{I}(N+1) - \mathbf{I}(N) = 0$. We extract a converging subsequence, such that $\lim \mathbf{I}(N'+1) - \mathbf{I}(N') = 0$. Hence we obtain (2.23) and (2.24) with $\delta = 0$. The equality case in Jensen's inequality yields $\alpha(s) \equiv \alpha$.

Asymptotic convergence (without any extraction). We have proved that there exists a subsequence such that $u_{N'}$ converges toward $u_\infty = h_\alpha$ in $L^2(0,T;B)$. We cannot pass to the limit pointwise in time from L^2 convergence. However, there exists a sequence of times $s_{N'} \in (0,1)$ such that $\|u_{N'}(s_{N'}) - u_\infty\|_B \to 0$. This includes uniform convergence and uniform decay at infinity. We can pass to the limit in the entropy term, and we obtain $H(u_{N'}(s_{N'})) \to H(u_\infty) = 0$. This means $H[n(N' + s_{N'})] \to 0$. Using the nonincreasing property of the entropy, we have $H[n(t)] \to 0$ as $t \to \infty$ (without extracting any subsequence).

Finally, we recall the Csiszar–Kullback inequality [15, 24]. For any nonnegative functions $f, g \in L^1(\mathbb{R}_+)$ such that $\int_{x>0} f(x)\,\mathrm{d}x = \int_{x>0} g(x)\,\mathrm{d}x = 1$, the following inequality holds true:

$$\|f - g\|^2_{L^1(\mathbb{R}_+)} \leq 4 \int_{x>0} f(x) \log\left(\frac{f(x)}{g(x)}\right)\,\mathrm{d}x.$$

This yields $\|n(t) - h_\alpha\|_{L^1} \to 0$.

2.2.2. Self-similar decay in the subcritical case. In the subcritical case $M < 1$ the density $n(t,z)$ is expected to decay with a self-similar diffusion scaling [7]. To catch this asymptotic behavior we rescale the density accordingly:

$$n(t,z) = \frac{1}{\sqrt{1+2t}} u\left(\log\sqrt{1+2t}, \frac{z}{\sqrt{1+2t}}\right).$$

The new density $u(\tau, y)$ satisfies

$$\partial_\tau u(\tau,y) = \partial_{yy} u(\tau,y) + \partial_y\left(y u(\tau,y)\right) + u(\tau,0)\partial_y u(\tau,y)\,, \tag{2.25}$$

together with a no-flux boundary condition: $\partial_y u(\tau,0) + u(\tau,0)^2 = 0$. The additional left-sided drift contributes to confine the mass in the new frame (τ, y). The unique stationary equilibrium in this new setting can be computed explicitly:

$$G_\alpha(y) = \alpha \exp\left(-\alpha y - y^2/2\right), \tag{2.26}$$

where α is uniquely defined by the condition $\int_{y>0} G_\alpha(y)\,\mathrm{d}y = M$. This rewrites $P(\alpha) = M$, P being an increasing function defined as follows:

$$P(\alpha) = \int_{y>0} \exp\left(-y - \frac{y^2}{2\alpha^2}\right)\,\mathrm{d}y, \quad \begin{cases} \lim_{\alpha\to 0} P(\alpha) = 0, \\ \lim_{\alpha\to+\infty} P(\alpha) = 1. \end{cases}$$

We redefine the relative entropy and the first momentum in the rescaled frame:

$$\mathbf{H}(\tau) = \int_{y>0} \frac{u(\tau,y)}{G_\alpha(y)} \log\left(\frac{u(\tau,y)}{G_\alpha(y)}\right) G_\alpha(y)\,\mathrm{d}y,$$

$$\mathbf{J}(\tau) = \int_{y>0} y u(\tau,y)\,\mathrm{d}y.$$

We also introduce a Lyapunov functional for (2.25):

$$\mathbf{L}(\tau) = \mathbf{H}(\tau) + \frac{1}{2(1-M)}\left(\mathbf{J}(\tau) - \alpha(1-M)\right)^2.$$

Note that it is a nonnegative quantity by Jensen's inequality.

LEMMA 2.7. *The Lyapunov functional* $\mathbf{L}$ *is nonincreasing:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}(\tau) = -\mathbf{D}(\tau) \le 0.$$

The dissipation reads as

$$\mathbf{D}(\tau) = \int_{y>0} u(\tau,y)\left(\partial_y \log u(\tau,y) + y + u(\tau,0)\right)^2\,\mathrm{d}y + \frac{1}{(1-M)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{J}(\tau)\right)^2. \tag{2.27}$$

Proof. We compute the evolution of the entropy as previously:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{H}(\tau) &= \int_{y>0} \partial_\tau u(\tau,y)\left(\log(u(\tau,y)) + \alpha y + \frac{y^2}{2}\right)\mathrm{d}y \\
&= -\int_{y>0} \left(\partial_y u(\tau,y) + u(\tau,0)u(\tau,y) + y u(\tau,y)\right)\left(\frac{\partial_y u(\tau,y)}{u(\tau,y)} + \alpha + y\right)\mathrm{d}y \\
&= -\int_{y>0} u(\tau,y)\left(\partial_y \log u(\tau,y) + y\right)^2\,\mathrm{d}y + u(\tau,0)^2 - u(\tau,0)\mathbf{J}(\tau) \\
&\quad + \alpha u(\tau,0) - \alpha\mathbf{J}(\tau) - \alpha u(\tau,0)M \\
&= -\int_{y>0} u(\tau,y)\left(\partial_y \log u(\tau,y) + y + u(\tau,0)\right)^2\,\mathrm{d}y \\
&\quad + (M-1)u(\tau,0)^2 + u(\tau,0)\mathbf{J}(\tau) + \alpha(1-M)u(\tau,0) - \alpha\mathbf{J}(\tau).
\end{aligned} \tag{2.28}$$

Moreover, the time evolution of the first momentum reads in the rescaled frame

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{J}(\tau) = (1-M)u(\tau,0) - \mathbf{J}(\tau).$$

As compared to (2.32), the additional contribution is due to the rescaling drift. We can eliminate $u(\tau,0)$ from (2.28) in the following two steps:

$$\begin{aligned}
u(\tau,0)\mathbf{J}(\tau) + \alpha(1-M)u(\tau,0) - \alpha\mathbf{J}(\tau) &= \frac{\mathbf{J}(\tau)}{(1-M)}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{J}(\tau) + \frac{\mathbf{J}(\tau)^2}{(1-M)} + \alpha\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{J}(\tau) \\
&= -\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\left(\mathbf{J}(\tau) - \alpha(1-M)\right)^2}{2(1-M)} \\
&\quad + \frac{2\mathbf{J}(\tau)}{(1-M)}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{J}(\tau) + \frac{\mathbf{J}(\tau)^2}{(1-M)}
\end{aligned} \tag{2.29}$$

and

$$-\frac{1}{(1-M)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{J}(\tau)\right)^2 = (M-1)u(\tau,0)^2 + \frac{2\mathbf{J}(\tau)}{(1-M)}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{J}(\tau) + \frac{\mathbf{J}(\tau)^2}{(1-M)}\,. \tag{2.30}$$

Combining (2.28)–(2.30), the proof of Lemma 2.7 is complete. ∎

To prove convergence of $u(\tau,\cdot)$ toward G_α we develop the same strategy as in section 2.2.1 for the critical case $M=1$. The main argument (apart from passing to the limit) consists in identifying the possible configurations u_∞ for which the dissipation $\mathbf{D}$ vanishes. In fact, this occurs if and only if both terms in (2.27) are zero. This means that $\mathbf{J}_\infty(\tau) = (1-M)u_\infty(\tau,0)$ on the one hand, and, on the other hand,

$$\partial_y \log u_\infty(\tau,y) + y + u_\infty(\tau,0) = 0.$$

We obtain that $u_\infty \equiv G_\alpha$, where G_α is given by (2.26). To pass to the limit as in section 2.2.1, we need to gain some good control of $\int_{y>0} u(\tau,y)\left(\partial_y \log u(\tau,y)\right)^2 \,\mathrm{d}y$ from the dissipation term $\mathbf{D}$. The situation here is simpler than in section 2.2.1 since the mass is subcritical. The argument goes as follows:

$$\begin{aligned}
&\int_{y>0} u(\tau,y)\left(\partial_y \log u(\tau,y) + y + u(\tau,0)\right)^2 \,\mathrm{d}y\\
&= \int_{y>0} u(\tau,y)\left(\partial_y \log u(\tau,y)\right)^2 \,\mathrm{d}y + (M-2)u(\tau,0)^2 + 2u(\tau,0)\mathbf{J}(\tau)\\
&\quad + \int_{y>0} y^2 u(\tau,y)\,\mathrm{d}y - 2M\\
&\geq \left(M + \frac{1}{M} - 2\right)u(\tau,0)^2 - 2M,
\end{aligned}$$

where we have used inequality (2.5). The quantity $M + M^{-1} - 2$ is positive since $M<1$. Hence, recalling Proposition 2.3, we can prove directly that $u(\cdot,0)$ belongs to L^2 locally in time (this was the purpose of (2.17)–(2.18)).

Finally, we obtain that $\mathbf{L}$ converges to zero as $\tau \to +\infty$. So $u(\tau,\cdot)$ converges toward G_α in the entropy sense.

2.3. Blow-up of solutions for supercritical mass. To prove that solutions blow up in finite time when mass is supercritical, $M>1$, and n_0 is nonincreasing, we show that the first momentum of $n(t,z)$ cannot remain positive for all time. This technique was first used by Nagai [27] and then by many authors in various contexts (see [5, 6, 14, 16, 13], for instance).

The assumption that n_0 is a nonincreasing function guarantees that $n(t,\cdot)$ is also nonincreasing for any time $t>0$ due to the maximum principle. In fact, the derivative $v(t,z) = \partial_z n(t,z)$ satisfies a parabolic-type equation without any source term, it is initially nonpositive, and it is nonpositive on the boundary due to (1.3).

Therefore, $-\partial_z n(t,z)/n(t,0)$ is a probability density at any time $t>0$. We deduce from the Jensen inequality the following interpolation estimate:

$$\left(\int_{z>0} z\frac{-\partial_z n(t,z)}{n(t,0)}\,\mathrm{d}z\right)^2 \leq \int_{z>0} z^2\frac{-\partial_z n(t,z)}{n(t,0)}\,\mathrm{d}z.$$

It rewrites in a more convenient way as

$$M^2 \leq 2n(t,0)\int_{z>0} zn(t,z)\,\mathrm{d}z. \tag{2.31}$$

We denote the first momentum $\mathbf{J}(t) = \int_{z>0} zn(t,z)\,\mathrm{d}z$. We plug (2.31) into the evolution of the moment (2.3):

$$\begin{aligned}\mathbf{J}(t) &= \mathbf{J}(0) + (1-M)\int_0^t n(s,0)\,\mathrm{d}s \\ &\le \mathbf{J}(0) + \frac{(1-M)M^2}{2}\int_0^t \frac{1}{\mathbf{J}(s)}\,\mathrm{d}s. \end{aligned} \tag{2.32}$$

We introduce the auxiliary function $\mathbf{K}(t) = \mathbf{J}(0) + (1-M)M^2\int_0^t \mathbf{J}(s)^{-1}\,\mathrm{d}s$. It is positive, and it satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{K}(t) = \frac{(1-M)M^2}{2}\frac{1}{\mathbf{J}(t)} \le \frac{(1-M)M^2}{2}\frac{1}{\mathbf{K}(t)}\,;$$

hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{K}(t)^2 \le (1-M)M^2\,.$$

We obtain a contradiction: the maximal time of existence T^* is necessarily finite when $M > 1$. On the other hand, following [23], it can be proved that the modulus of integrability has to become singular at T^*:

$$\lim_{K\to+\infty}\left(\sup_{t\in(0,T^*)}\int_{z>0}(n(t,z)-K)_+\,\mathrm{d}z\right) > 0.$$

Otherwise a truncation method enables us to prove local existence by replacing n with $(n-K)_+$ for K sufficiently large.

Remark 2.8. It is natural to perform the Laplace transform on the equation (2.1): $\mathcal{L}_z(n(t,z)) = \hat{n}(t,\zeta) = \int_{z>0} n(t,z)\exp(-\zeta z)\,\mathrm{d}z$. Then the occurrence of blow-up is clear after transformation. We refer the reader to [8], where the Fourier transformation has been applied successfully to analyzing a one-dimensional caricature of the two-dimensional Keller–Segel equation.

3. Variants of blow-up criteria. In this section we determine necessary conditions for blow-up to occur for a fast decaying interaction potential (section 3.1) and for a finite interval (section 3.2).

3.1. Finite range of action. In this part we consider the system

$$\partial_t n(t,z) = \partial_{zz} n(t,z) - \partial_z\left(n(t,z)\partial_z\phi(t,z)\right), \quad t>0\,,\ z\in(0,+\infty), \tag{3.1}$$

with zero-flux at $z=0$, and the attractive potential is given by

$$-\partial_{zz}\phi(t,z) + \alpha^2\phi(t,z) = 0\,, \quad -\partial_z\phi(t,0) = n(t,0)\,. \tag{3.2}$$

We introduce the exponential moment of the solution:

$$\mathbf{J}_\alpha(t) = \int_{z>0}\exp(\alpha z)n(t,z)\,\mathrm{d}z.$$

PROPOSITION 3.1. *Assume $M>1$ and the exponential moment is small in the sense of criterion* (3.3) *below. Assume in addition that* $\exp(-\alpha z)n_0(z)$ *is a nonincreasing function. Then the solution to* (3.1)–(3.2) *with initial data* $n(0,z) = n_0(z)$ *blows up in finite time.*

Proof. The attractive field is given by $\partial_z \phi(t,z) = -\exp(-\alpha z) n(t,0)$. Similarly to the proof of Theorem 1.2, we compute the time derivative of $\mathbf{J}_\alpha(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{J}_\alpha(t) = \alpha^2 \mathbf{J}_\alpha(t) + \alpha n(t,0)(1-M)\,.$$

We check that the function $u(t,z) = \exp(-\alpha z)n(t,z)$ is decreasing w.r.t. z for all time $t > 0$. For this purpose we write the equation for $v(t,z) = \partial_z u(t,z)$. This reads as

$$\begin{aligned}\partial_t u(t,z) &= \partial_{zz} u(t,z) + 2\alpha \partial_z u(t,z) + \alpha^2 u(t,z) + \exp(-\alpha z) n(t,0) \partial_z u(t,z),\\ \partial_t v(t,z) &= \partial_{zz} v(t,z) + 2\alpha \partial_z v(t,z) + \alpha^2 v(t,z) + \exp(-\alpha z) n(t,0) \partial_z v(t,z)\\ &\quad -\alpha \exp(-\alpha z) n(t,0) v(t,z).\end{aligned}$$

Since the boundary condition reads $v(t,0) = -\alpha n(t,0) - n(t,0)^2 \le 0$ and the above parabolic equation preserves nonpositivity, we deduce that $v(t,z) \le 0$ if $v(0,z) \le 0$.

We can adapt the inequality (2.31) to the function $u(t,z)$, and we obtain

$$\begin{aligned} M^4 &\le \left(\int_{z>0} \exp(\alpha z) n(t,z)\,\mathrm{d}z \right)^2 \left(\int_{z>0} u(t,z)\,\mathrm{d}z \right)^2 \\ &\le \mathbf{J}_\alpha(t)^2 n(t,0)^2 \left(\int_{z>0} z \frac{-\partial_z u(t,z)}{u(t,0)}\,\mathrm{d}z \right)^2 \\ &\le \mathbf{J}_\alpha(t)^2 n(t,0)^2 \int_{z>0} \left(\frac{\exp(2\alpha z) - 1 - 2\alpha z}{2\alpha^2} \right) \frac{-\partial_z u(t,z)}{u(t,0)}\,\mathrm{d}z \\ &\le \frac{1}{\alpha} \mathbf{J}_\alpha(t)^2 n(t,0) \int_{z>0} \left(\exp(\alpha z) - \exp(-\alpha z)\right) n(t,z)\,\mathrm{d}z \\ &\le \frac{1}{\alpha} \mathbf{J}_\alpha(t)^2 n(t,0) \left(\mathbf{J}_\alpha(t) - \frac{M^2}{\mathbf{J}_\alpha(t)} \right). \end{aligned}$$

Finally, when $M > 1$ we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{J}_\alpha(t) \le \alpha^2 \mathbf{J}_\alpha(t) + \frac{\alpha^2 (1-M) M^4}{\mathbf{J}_\alpha(t)^3 \left(1 - \frac{M^2}{\mathbf{J}_\alpha(t)^2}\right)}\,.$$

Notice that $\mathbf{J}_\alpha(0) > M$ by definition. We get an obstruction to global existence if the following condition holds true:

$$\frac{\mathbf{J}_\alpha(0)^4}{M^4} \left(1 - \frac{M^2}{\mathbf{J}_\alpha(0)^2} \right) < (M-1)\,. \qquad \square \tag{3.3}$$

3.2. Finite interval. In this part we consider (1.2) on a finite interval $(0,L)$ for some $L > 0$, namely,

$$\partial_t n(t,z) = \partial_{zz} n(t,z) + (n(t,0) - n(t,L)) \partial_z n(t,z)\,, \quad t > 0\,,\ x \in (0,L)\,, \tag{3.4}$$

together with $n(t=0,z) = n_0(z) \ge 0$ and zero-flux boundary conditions at both sides of the interval.

Equilibrium configurations are given by the family of functions

$$h(z) = \alpha \exp(-(\alpha-\beta)z)\,, \quad \beta = \alpha \exp(-(\alpha-\beta)L)\,. \tag{3.5}$$

There are two possibilities; either $\alpha = \beta$ and h is constant, or $\alpha \neq \beta$ and $M = \int_0^L h(z)\,\mathrm{d}z = 1$. Observe that given $\alpha > 0$ there exists a unique β satisfying (3.5). If $\alpha L < 1$, then $\beta > \alpha$ (h is increasing), whereas if $\alpha L > 1$, then $\beta < \alpha$ (h is decreasing).

PROPOSITION 3.2. *Assume $M > 1$ and the first moment is small: $4\mathbf{J}(0) < LM$. Assume in addition that $n_0(z)$ is a nonincreasing function. Then the solution to* (3.4) *with initial data $n(0,z) = n_0(z)$ blows up in finite time.*

Proof. We proceed again as in the proof of Theorem 1.2. From Jensen's inequality, it follows that

$$\left(\int_0^L z\frac{-\partial_z n(t,z)}{n(t,0)-n(t,L)}\,\mathrm{d}z\right)^2 \leq \int_0^L z^2\frac{-\partial_z n(t,z)}{n(t,0)-n(t,L)}\,\mathrm{d}z;$$

hence, using that $n(t,0) > n(t,L)$ for any time $t > 0$, we deduce that

$$(M - Ln(t,L))^2 \leq (n(t,0)-n(t,L))\left(2\int_0^L zn(t,z)\,\mathrm{d}z - L^2 n(t,L)\right), \tag{3.6}$$

and the inequality remains true when $n(t,0) = n(t,L)$ and $n(t,\cdot)$ is constant. Therefore, the first momentum $\mathbf{J}(t) = \int_0^L zn(t,z)\,\mathrm{d}z$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{J}(t) &= (1-M)(n(t,0)-n(t,L))\\ &\leq (1-M)\frac{(M-Ln(t,L))^2}{2\mathbf{J}(t)-L^2n(t,L)}\\ &\leq (1-M)\frac{M^2-2MLn(t,L)}{2\mathbf{J}(t)}\,.\end{aligned}$$

On the other hand, from (3.6) again, it follows that $2\mathbf{J}(t) \geq L^2 n(t,L)$, and we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{J}(t) \leq \frac{M(1-M)}{2\mathbf{J}(t)}\left(M - \frac{4\mathbf{J}(t)}{L}\right),$$

and the result follows by contradiction as in section 2.3. ∎

4. The model with dynamical exchange of markers at the boundary: Prevention of blow-up and asymptotic behavior. In section 2.3, we proved that finite blow-up occurs in the basic model (2.1) when mass is supercritical, $M > 1$. On the other hand, the model which was originally proposed in [18] is

$$\begin{cases}\partial_t n(t,z) = \partial_{zz} n(t,z) + \mu(t)\partial_z n(t,z)\,, \quad t > 0\,,\, z \in (0,+\infty)\,,\\ \frac{\mathrm{d}}{\mathrm{d}t}\mu(t) = n(t,0) - \mu(t)\,,\end{cases}$$

together with the flux condition at the boundary:

$$\partial_z n(t,0) + \mu(t)n(t,0) = \frac{\mathrm{d}}{\mathrm{d}t}\mu(t)\,. \tag{4.1}$$

The quantity μ represents the concentration of markers which are stuck to the boundary and thus create the attracting drift. The dynamics of μ is driven by simple attachment/detachment kinetics. The mass of molecular markers is shared between the free particles $n(t,z)$ and the particles on the boundary $\mu(t)$. The boundary condition (4.1) guarantees conservation of the total mass:

$$\int_{z>0} n(t,z)\,\mathrm{d}z + \mu(t) = M. \tag{4.2}$$

From (4.2), we easily deduce that finite time blow-up cannot occur since the drift $\mu(t)$ is bounded by M. We denote by $m(t)$ the mass of free particles:

$$m(t) = \int_{z>0} n(t,z)\,\mathrm{d}z.$$

The conservation of mass reads

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) + \frac{\mathrm{d}}{\mathrm{d}t}\mu(t) = 0.$$

We redefine the relative entropy as follows:

$$\mathbf{H}(t) = \int_{z>0} \frac{n(t,z)}{m(t)h(z)} \log\left(\frac{n(t,z)}{m(t)h(z)}\right) h(z)\,\mathrm{d}z,$$

where the asymptotic profile h is given by

$$h(z) = \nu \exp\left(-\nu z\right), \quad \nu = M - 1.$$

When mass is supercritical, $M > 1$, we shall prove that the density of free markers $n(t,z)$ converges in relative entropy toward h, whereas the concentration of markers stuck at the boundary $\mu(t)$ converges to ν. This is achieved using a suitable Lyapunov functional as in sections 2.2.1 and 2.2.2. We introduce accordingly

$$\mathbf{L}(t) = m(t)\mathbf{H}(t) + \frac{1}{2}\left(\mu(t) - \nu\right)^2 + \mu(t)\log\left(\frac{\mu(t)}{\nu}\right) + m(t)\log m(t).$$

The rest of this section is devoted to the proof of the following lemma.

LEMMA 4.1. *The Lyapunov functional* $\mathbf{L}$ *is nonincreasing:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}(t) = -\mathbf{D}(t) \le 0\,.$$

The dissipation reads as follows:

$$\mathbf{D}(t) = \int_{z>0} n(t,z)\left(\partial_z \log n(t,z) + \frac{n(t,0)}{m(t)}\right)^2 \mathrm{d}z + m(t)\left(\frac{n(t,0)}{m(t)} - \mu(t)\right)^2$$
$$+ \left(n(t,0) - \mu(t)\right)\log\left(\frac{n(t,0)}{\mu(t)}\right) + \mu(t)\left(\mu(t) - \nu\right)^2\,.$$

Proof. We compute below the time evolution of the relative entropy. This is strongly inspired by the previous computation, but this takes into consideration the nonconservation of mass for the free markers' density and the additional dynamics of μ.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(m(t)\mathbf{H}(t)\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{z>0} n(t,z)\left(\log\left(\frac{n(t,z)}{m(t)}\right) - \log\nu + \nu z\right)\mathrm{d}z \\
&= \int_{z>0} \partial_t\left(n(t,z)\right)\left(\log\left(\frac{n(t,z)}{m(t)}\right) - \log\nu + \nu z\right)\mathrm{d}z \\
&\quad + \int_{z>0} n(t,z)\,\partial_t \log\left(\frac{n(t,z)}{m(t)}\right)\mathrm{d}z \\
&= \int_{z>0} \partial_z\left(\partial_z n(t,z) + \mu(t)n(t,z)\right)\left(\log\left(\frac{n(t,z)}{m(t)}\right) + \nu z\right)\mathrm{d}z \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t} m(t)\log\nu,
\end{aligned}$$

where we have used the identity

$$\int_{z>0} n(t,z)\,\partial_t \log\left(\frac{n(t,z)}{m(t)}\right)\mathrm{d}z = m(t)\int_{z>0}\partial_t\left(\frac{n(t,z)}{m(t)}\right)\mathrm{d}z = 0.$$

We integrate by parts to get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(m(t)\mathbf{H}(t) + m(t)\log\nu\right) &= -\int_{z>0}\left(\partial_z n(t,z) + \mu(t)n(t,z)\right)\left(\frac{\partial_z n(t,z)}{n(t,z)} + \nu\right)\mathrm{d}z \\
&\quad - \left(\partial_z n(t,0) + \mu(t)n(t,0)\right)\log\left(\frac{n(t,0)}{m(t)}\right) \\
&= -\int_{z>0} n(t,z)\left(\partial_z \log n(t,z)\right)^2 \mathrm{d}z + (\nu + \mu(t))n(t,0) \\
&\quad - m(t)\mu(t)\nu - \log\left(\frac{n(t,0)}{m(t)}\right)\frac{\mathrm{d}}{\mathrm{d}t}\mu(t).
\end{aligned}$$

We again use the following key identity:

$$\int_{z>0} n(t,z)\left(\partial_z \log n(t,z)\right)^2 \mathrm{d}z = \int_{z>0} n(t,z)\left(\partial_z \log n(t,z) + \frac{n(t,0)}{m(t)}\right)^2 \mathrm{d}z + \frac{n(t,0)^2}{m(t)}.$$

We end up with the following expression for the dissipation of the corrected entropy:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(m(t)\mathbf{H}(t) + m(t)\log\nu\right) &= -\int_{z>0} n(t,z)\left(\partial_z \log n(t,z) + \frac{n(t,0)}{m(t)}\right)^2 \mathrm{d}z \\
&\quad - \frac{n(t,0)^2}{m(t)} + (\nu + \mu(t))n(t,0) - m(t)\mu(t)\nu \\
&\quad - \log\left(\frac{n(t,0)}{m(t)}\right)\frac{\mathrm{d}}{\mathrm{d}t}\mu(t)\,.
\end{aligned}$$

On one hand, we have that

$$\begin{aligned}
-\frac{n(t,0)^2}{m(t)} + (\nu + \mu(t))n(t,0) - m(t)\mu(t)\nu &= \left(-\frac{n(t,0)}{m(t)} + \nu\right)\left(n(t,0) - m(t)\mu\right) \\
&= -m(t)\left(\frac{n(t,0)}{m(t)} - \mu(t)\right)^2 \\
&\quad - (\mu(t) - \nu)\left(n(t,0) - m(t)\mu(t)\right),
\end{aligned}$$

and, on the other hand, we see that

$$
\begin{aligned}
&-\log\left(\frac{n(t,0)}{m(t)}\right)\frac{\mathrm{d}}{\mathrm{d}t}\mu(t)\\
&\quad= -\log\left(\frac{n(t,0)}{\mu(t)}\right)\frac{\mathrm{d}}{\mathrm{d}t}\mu(t) - \log\left(\frac{\mu(t)}{m(t)}\right)\frac{\mathrm{d}}{\mathrm{d}t}\mu(t)\\
&\quad= -\left(n(t,0)-\mu(t)\right)\log\left(\frac{n(t,0)}{\mu(t)}\right) - \log\left(\mu(t)\right)\frac{\mathrm{d}}{\mathrm{d}t}\mu(t) - \log\left(m(t)\right)\frac{\mathrm{d}}{\mathrm{d}t}m(t)\\
&\quad= -\left(n(t,0)-\mu(t)\right)\log\left(\frac{n(t,0)}{\mu(t)}\right)\\
&\qquad -\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)\log\mu(t) - \mu(t) + m(t)\log m(t) - m(t) - \nu\log\nu + M\right).
\end{aligned}
$$

The last contribution to be reformulated is

$$
\begin{aligned}
-(\mu(t)-\nu)\left(n(t,0)-m(t)\mu(t)\right) &= -(\mu(t)-\nu)\left(\frac{\mathrm{d}}{\mathrm{d}t}\mu(t) + (1-m(t))\mu(t)\right)\\
&= -(\mu(t)-\nu)\left(\frac{\mathrm{d}}{\mathrm{d}t}\mu(t) + (\mu(t)-\nu)\mu(t)\right)\\
&= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)-\nu)^2 - \mu(t)(\mu(t)-\nu)^2\,.
\end{aligned}
$$

Combining all these calculations we conclude the proof of Lemma 4.1. ◻

Following the reasoning of section 2.2.1, we can prove that $\mu(t)$ converges to ν, the partial mass $m(t)$ converges to 1, and the density $n(t,\cdot)$ converges to the stationary state h as $t\to\infty$. We omit the details.

5. The higher-dimensional case $N \geq 2$. In this section we investigate the possible behaviors of (1.1) in dimension $N \geq 2$ with the two possible choices (1.5) and (1.6) for the advection field.

5.1. Global existence. We give the proof of Theorem 1.6. Since many of the arguments are similar to the one-dimensional case, we only sketch the proof and focus on the propagation of L^p bounds, which is the crucial a priori estimate as soon as entropy methods are lacking [23].

Let n be a solution of (1.1) with $\nabla\cdot\mathbf{u}\geq 0$ and $\mathbf{u}(t,y,0)\cdot\mathbf{e}_z = n(t,y,0)$. We see that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{H}} n(t,x)^p\,\mathrm{d}x &= -p\int_{\mathcal{H}}\nabla n(t,x)^{p-1}\cdot\nabla n(t,x)\,\mathrm{d}x\\
&\quad + p\int_{\mathcal{H}}\nabla n(t,x)^{p-1}\cdot\mathbf{u}(t,x)\,n(t,x)\,\mathrm{d}x.
\end{aligned}
\tag{5.1}
$$

On one hand, we have that

$$
-p\int_{\mathcal{H}}\nabla n(t,x)^{p-1}\cdot\nabla n(t,x)\,\mathrm{d}x = -\frac{4(p-1)}{p}\int_{\mathcal{H}}\left|\nabla n(t,x)^{p/2}\right|^2\,\mathrm{d}x,
$$

and, on the other hand,

$$
\frac{p}{p-1}\int_{\mathcal{H}}\nabla n(t,x)^{p-1}\cdot\mathbf{u}(t,x)\,n(t,x)\,\mathrm{d}x = -\int_{\mathcal{H}} n(t,x)^p\left(\nabla\cdot\mathbf{u}\right)\,\mathrm{d}x + \int_{x} n(t,y,0)^{p+1}\,\mathrm{d}y.
$$

To estimate the two opposite trends in (5.1) we use the following Sobolev trace inequality [4, 28]: there exists a constant C_r such that for any nonnegative $f \in W^{1,r}$ we have

$$\left(\int_{y\in\mathbb{R}^{N-1}} f(y,0)^{r^*}\,\mathrm{d}y \right)^{1/r^*} \le C_r \left(\int_{\mathcal{H}} |\nabla f(x)|^r\,\mathrm{d}x \right)^{1/r}, \tag{5.2}$$

where $r^* = \frac{(N-1)r}{N-r}$. Applying the previous inequality (5.2) with $f = n^s$, we obtain the estimates

$$\begin{aligned} \int_{y\in\mathbb{R}^{N-1}} n(t,y,0)^{r^* s}\,\mathrm{d}y &\le C_r \left(\frac{2s}{p}\right)^{r^*} \left(\int_{\mathcal{H}} \left|\nabla n(t,x)^{p/2}\, n(t,x)^{s-\frac{p}{2}}\right|^r \mathrm{d}x \right)^{r^*/r} \\ &\le C_r \left(\frac{2s}{p}\right)^{r^*} \left(\int_{\mathcal{H}} \left|\nabla n(t,x)^{p/2}\right|^2 \mathrm{d}x \right)^{r^*/2} \left(\int_{\mathcal{H}} \left(n(t,x)^{s-\frac{p}{2}}\right)^{\frac{2r}{2-r}} \mathrm{d}x \right)^{\frac{(2-r)r^*}{2r}}. \end{aligned}$$

We infer that L^N is the critical space for global existence. Hence we choose

$$\left(s - \frac{p}{2}\right)\frac{2r}{2-r} = N.$$

On the other hand, we also require that

$$1 = \frac{r^*}{2} = \frac{1}{2}\frac{(N-1)r}{N-r}.$$

A straightforward computation leads to

$$r = \frac{2N}{N+1}, \quad s = \frac{p+1}{2}, \quad r^* s = p+1, \quad \frac{(2-r)r^*}{2r} = \frac{1}{N}.$$

Therefore, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{H}} n(t,x)^p\,\mathrm{d}x \le -\frac{4(p-1)}{p}\left(1 - C\|n(t)\|_{L^N}\right)\int_{\mathcal{H}} \left|\nabla n(t,x)^{p/2}\right|^2 \mathrm{d}x.$$

The peculiar choice $p = N$ yields global existence if $\|n(0)\|_{L^N}$ is smaller than some explicit threshold as in [14].

5.2. Blow-up of solutions in the first case (1.5). We compute the evolution of the second momentum $\mathbf{I}(t) = \frac{1}{2}\int_{\mathcal{H}} |x|^2 n(t,x)\,\mathrm{d}x$ as for the classical Keller–Segel system (see [30] and references therein):

$$\frac{\mathrm{d}\mathbf{I}(t)}{\mathrm{d}t} = NM - \int_{\mathcal{H}} z n(t,y,0) n(t,y,z)\,\mathrm{d}x.$$

Next, define $M(t,y) = \int_{z>0} n(t,y,z)\,\mathrm{d}z$. Under the assumption $\partial_z n(t,x) \le 0$ for all $x \in \mathcal{H}$ and $t > 0$, inequality (2.31) rewrites as

$$M(t,y)^2 \le 2n(t,y,0)\int_{z>0} z n(t,y,z)\,\mathrm{d}z.$$

We deduce that

$$\frac{\mathrm{d}\mathbf{I}(t)}{\mathrm{d}t} \le NM - \frac{1}{2}\|M(t,y)\|_{L^2}^2. \tag{5.3}$$

By interpolation there exists a constant C such that

$$M^{\frac{N+3}{2}} \le C\mathbf{I}(t)^{\frac{N-1}{2}} \|M(t,y)\|_{L^2}^2 . \tag{5.4}$$

Indeed, we have

$$\begin{aligned} M &= \int_{|y|<R} M(t,y)\,\mathrm{d}y + \int_{|y|>R} M(t,y)\,\mathrm{d}y \\ &\le CR^{(N-1)/2}\left(\int_{R^{N-1}} M(t,y)^2\,\mathrm{d}y\right)^{1/2} + R^{-2}\int_{R^{N-1}} |y|^2 M(t,y)\,\mathrm{d}y \\ &\le CR^{(N-1)/2}\|M(t,y)\|_{L^2} + R^{-2}\mathbf{I}(t). \end{aligned}$$

Optimizing with respect to R, we get (5.4). Combining (5.4) and (5.3), we conclude that the solution blows up in finite time if $\mathbf{I}(0) \le CM^{\frac{N+1}{N-1}}$.

5.3. Blow-up in the second case (1.6). We recall the expression of the advection field in the potential case (1.6):

$$\mathbf{u}(t,x) = -\int_{y'\in\mathbb{R}^{N-1}} \frac{(y-y',z)}{\left(|y-y'|^2+z^2\right)^{N/2}} n(t,y',0)\,\mathrm{d}y'.$$

Therefore, we have

$$\begin{aligned} \frac{\mathrm{d}\mathbf{I}(t)}{\mathrm{d}t} &= NM + \int_{\mathcal{H}} x\cdot(n(t,x)\mathbf{u}(t,x))\,\mathrm{d}x \\ &= NM - \iint_{y,y'}\int_{z>0} \frac{y\cdot(y-y')+z^2}{\left(|y-y'|^2+z^2\right)^{N/2}} n(t,y',0)n(t,y,z)\,\mathrm{d}y\,\mathrm{d}y'\,\mathrm{d}z. \end{aligned}$$

We use a symmetrization trick to evaluate the contribution of interaction:

$$\begin{aligned} &\iint_{y,y'}\int_{z>0} \frac{y\cdot(y-y')}{\left(|y-y'|^2+z^2\right)^{N/2}} n(t,y',0)n(t,y,z)\,\mathrm{d}y\,\mathrm{d}y'\,\mathrm{d}z \\ &= \frac{1}{2}\iint_{y,y'}\int_{z>0} \frac{y-y'}{\left(|y-y'|^2+z^2\right)^{N/2}} \\ &\quad\cdot\left(n(t,y',0)n(t,y,z)y - n(t,y,0)n(t,y',z)y'\right)\,\mathrm{d}y\,\mathrm{d}y'\,\mathrm{d}z. \end{aligned}$$

LEMMA 5.1. *Let f be a smooth positive function. Assume that we have both $\partial_z f(x) \le 0$ and*

$$\forall z>0\,,\ \forall y\in\mathbb{R}^{N-1}\,,\ \forall h\in\mathbb{R}^{N-1}\quad (h\cdot y)\left(h\cdot\partial_z\nabla_y \log f(x)\right) \ge 0. \tag{5.5}$$

Then for all $y, y' \in \mathbb{R}^{N-1}$ and for all $z > 0$, the following inequality holds true:

$$(y-y')\cdot\left(f(y',0)f(y,z)y - f(y,0)f(y',z)y'\right) \ge |y-y'|^2 f(y,z)f(y',z). \tag{5.6}$$

Proof. Inequality (5.6) rewrites as

$$(y-y')\cdot\left(\frac{f(y,z)}{f(y,0)}\left(1-\frac{f(y',z)}{f(y',0)}\right)y - \frac{f(y',z)}{f(y',0)}\left(1-\frac{f(y,z)}{f(y,0)}\right)y'\right) \ge 0.$$

Since $\partial_z f(x) \le 0$ we have both $f(y,z) \le f(y,0)$ and $f(y',z) \le f(y',0)$ for all y, y', z. Hence we are reduced to proving that the vector field

$$\frac{\frac{f(y,z)}{f(y,0)}}{1 - \frac{f(y,z)}{f(y,0)}} y$$

is monotonic with respect to the y variable. Computing the derivative with respect to y, it is straightforward to check that it is monotonic if (5.5) is satisfied,

$$\begin{aligned}
\nabla_y \left(\frac{\frac{f(y,z)}{f(y,0)}}{1 - \frac{f(y,z)}{f(y,0)}} y \right) &= \left(\frac{\frac{f(y,z)}{f(y,0)}}{\left(1 - \frac{f(y,z)}{f(y,0)}\right)^2} \right) \left(\frac{\nabla_y f(y,z)}{f(y,z)} - \frac{\nabla_y f(y,0)}{f(y,0)} \right) \otimes y \\
&\quad + \left(\frac{f(y,z)}{f(y,0) - f(y,z)} \right) \mathrm{Id} \\
&\ge \left(\frac{\frac{f(y,z)}{f(y,0)}}{\left(1 - \frac{f(y,z)}{f(y,0)}\right)^2} \right) \left(\int_{z'=0}^{z} \partial_z \nabla_y \log f(y,z') \,\mathrm{d}z' \right) \otimes y \ge 0 \,,
\end{aligned}$$

in the following matrix sense: $A^T + A \ge 0$. ◻

Under the hypotheses of Theorem 1.7 we assume that conditions (5.5)–(5.6) are fulfilled for every time of existence. We deduce that

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{I}(t)}{\mathrm{d}t} &\le NM - \frac{1}{2} \iint_{y,y'} \int_{z>0} \frac{|y-y'|^2 + 2z^2}{\left(|y-y'|^2 + z^2\right)^{N/2}} n(t,y',z) n(t,y,z) \,\mathrm{d}y \,\mathrm{d}y' \,\mathrm{d}z \\
&\le NM - \frac{1}{2} \iint_{y,y'} \int_{z>0} \frac{1}{\left(|y-y'|^2 + z^2\right)^{N/2-1}} n(t,y',z) n(t,y,z) \,\mathrm{d}y \,\mathrm{d}y' \,\mathrm{d}z.
\end{aligned}$$

Since $|y-y'|^2 + z^2 \le 2|y|^2 + 2|y'|^2 + z^2$, and n is nonnegative, we have

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{I}(t)}{\mathrm{d}t} &\le NM - \frac{1}{2} \iiint_{\left\{|y|<\frac{R}{3}, |y'|<\frac{R}{3}, z<\frac{2R}{3}\right\}} R^{2-N} n(t,y',z) n(t,y,z) \,\mathrm{d}y \,\mathrm{d}y' \,\mathrm{d}z \\
&\le NM - \frac{R^{2-N}}{2} \int_{0<z<\frac{2R}{3}} \left(\int_{|y|<\frac{R}{3}} n(t,y,z) \,\mathrm{d}y \right)^2 \mathrm{d}z \\
&\le NM - \frac{3R^{1-N}}{4} \left(\int_{0<z<\frac{2R}{3}} \int_{|y|<\frac{R}{3}} n(t,y,z) \,\mathrm{d}y \,\mathrm{d}z \right)^2 ,
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality. We have therefore

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{I}(t)}{\mathrm{d}t} &\le NM - \frac{3R^{1-N}}{4} \left(M - \iint_{\left\{z>\frac{2R}{3} \text{ or } |y|>\frac{R}{3}\right\}} n(t,y,z) \,\mathrm{d}y \,\mathrm{d}z \right)^2 \\
&\le NM - \frac{R^{1-N}}{2} M^2 + CR^{-N-3} \mathbf{I}(t)^2 \,,
\end{aligned}$$

because $R^2 < 9|x|^2$ on $\left\{z > \frac{2R}{3} \text{ or } |y| > \frac{R}{3}\right\}$. Optimizing with respect to R, we conclude that the solution blows up in finite time if $\mathbf{I}(0) \le CM^{\frac{N+1}{N-1}}$, similarly as in section 5.2.

6. Conclusion. In this work we have made some progress in the nonlinear analysis of the model introduced in [18]. We have demonstrated that this model and some variants exhibit pattern formation (either blow-up or convergence toward a nonhomogeneous steady state) under some conditions. However, we have not answered the main question: do they describe cell polarization or not? Analyzing the one-dimensional case is informative in understanding whether active transport is strong enough to drive the molecular markers toward the boundary. When the interval is finite, it describes a one-dimensional cell which can polarize spontaneously from a homogeneous distribution of markers (see Figure 1.3(a) and section 3.2). Unfortunately the analysis of the higher-dimensional case is more difficult. Obviously the first model (1.5) does not exhibit cell polarization since we can integrate (1.1) with respect to z, and we obtain for $\nu(t,y) = \int_{z>0} n(t,y,z)\,\mathrm{d}z$

$$\partial_t \nu(t,y) = \partial_{yy}\nu(t,y)\,.$$

Thus there is no transversal instability, which is the main feature of spontaneous cell polarization. This confirms the linear analysis performed in [18]. On the other hand, the second model (1.6) is expected to develop symmetry breaking in dimension $N = 2$ since the tangential component of the advective field on the boundary is given by the Hilbert transform of the trace $n(t, y, 0)$ which is known to enhance finite time aggregation [11]. However, there is no clear mathematical distinction between the two models as continuation after the blow-up time appears to be very delicate in a similar context [33, 34, 17]. It would be very interesting to make such a distinction beyond linear analysis as performed in [18]. We leave it for further work.

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REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
[2] B. Alberts, *Molecular Biology of the Cell*, 4th ed., Garland Science, New York, 2002.
[3] J.-P. Aubin, *Un théorème de compacité*, C. R. Acad. Sci. Paris, 256 (1963), pp. 5042–5044.
[4] R. J. Biezuner, *Best constants in Sobolev trace inequalities*, Nonlinear Anal., 54 (2003), pp. 575–589.
[5] P. Biler, *Existence and nonexistence of solutions for a model of gravitational interaction of particle* III, Colloq. Math., 68 (1995), pp. 229–239.
[6] P. Biler and W. A. Woyczyński, *Global and exploding solutions for nonlocal quadratic evolution problems*, SIAM J. Appl. Math., 59 (1998), pp. 845–869.
[7] A. Blanchet, J. Dolbeault, and B. Perthame, *Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations, 44 (2006).
[8] V. Calvez and J. A. Carrillo, *Refined asymptotics for the subcritical Keller-Segel system and related functional inequalities*, Proc. Amer. Math. Soc., to appear.
[9] V. Calvez, L. Corrias, and A. Ebde, *Blow-up, concentration phenomenon and global existence for the Keller-Segel model in high dimension*, Comm. Partial Differential Equations, 37 (2012), pp. 561–584.
[10] V. Calvez, N. Meunier, and R. Voituriez, *A one-dimensional Keller-Segel equation with a drift issued from the boundary boundary*, C. R. Math. Acad. Sci. Paris, 348 (2010), pp. 629–634.
[11] V. Calvez, B. Perthame, and M. Sharifi tabar, *Modified Keller-Segel system and critical mass for the log interaction kernel*, in Nonlinear Partial Differential Equations and Related Analysis, Contemp. Math. 429, AMS, Providence, RI, 2007.

[12] A. E. Carlsson, *Dendritic actin filament nucleation causes traveling waves and patches*, Phys. Rev. Lett., 104 (2010), 228102.
[13] T. Cieślak and P. Laurençot, *Finite time blow-up for a one-dimensional quasilinear parabolic-parabolic chemotaxis system*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), pp. 437–446.
[14] L. Corrias, B. Perthame, and H. Zaag, *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan J. Math., 72 (2004), pp. 1–29.
[15] I. Csiszár, *Information-type measures of difference of probability distributions and indirect observations*, Studia Sci. Math. Hungar., 2 (1967), pp. 299–318.
[16] J. Dolbeault and B. Perthame, *Optimal critical mass in the two-dimensional Keller-Segel model in* $\mathbb{R}^2$, C. R. Math. Acad. Sci. Paris, 339 (2004), pp. 611–616.
[17] J. Dolbeault and C. Schmeiser, *The two-dimensional Keller-Segel model after blow-up*, Discrete Contin. Dyn. Syst., 25 (2009), pp. 109–121.
[18] R. J. Hawkins, O. Bénichou, M. Piel, and R. Voituriez, *Rebuilding cytoskeleton roads: Active transport induced polarisation of cells*, Phys. Rev. E, 80 (2009), 040903.
[19] M. A. Herrero and J. J. L. Velázquez, *Singularity formation in the one-dimensional supercooled Stefan problem*, European J. Appl. Math., 7 (1996), pp. 119–150.
[20] M. A. Herrero and J. J. L. Velázquez, *Chemotactic collapse for the Keller-Segel model*, J. Math. Biol., 35 (1996), pp. 177–194.
[21] P. A. Iglesias and P. N. Devreotes, *Navigating through models of chemotaxis*, Curr. Opin. Cell Biol., 20 (2008), pp. 35–40.
[22] J. E. Irazoqui, A. S. Howell, C. L. Theesfeld, and D. J. Lew, *Opposing roles for actin in Cdc42p polarisation*, Mol. Biol. Cell, 16 (2005), pp. 1296–1304.
[23] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc., 329 (1992), pp. 819–824.
[24] S. Kullback, *On the convergence of discrimination information*, IEEE Trans. Inform. Theory, 14 (1968), pp. 765–766.
[25] H. Levine, D. A. Kessler, and W.-J. Rappel, *Directional sensing in eukaryotic chemotaxis: A balanced inactivation model*, Proc. Natl. Acad. Sci. USA, 103 (2006), pp. 9761–9766.
[26] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.
[27] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci., 5 (1995), pp. 581–601.
[28] B. Nazaret, *Best constant in Sobolev trace inequalities on the half-space*, Nonlinear Anal., 65 (2006), pp. 1977–1985.
[29] M. Onsum and C. V. Rao, *A mathematical model for neutrophil gradient sensing and polarisation*, PLoS Comput. Biol., 3 (2007), e36.
[30] B. Perthame, *Transport Equations in Biology*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2007.
[31] R. B. Phillips, J. Kondev, and J. Theriot, *Physical Biology of the Cell*, Garland Science, New York, 2009.
[32] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl., 146 (1987), pp. 65–96.
[33] J. J. L. Velázquez, *Point dynamics in a singular limit of the Keller–Segel model* 1: *Motion of the concentration regions*, SIAM J. Appl. Math., 64 (2004), pp. 1198–1223.
[34] J. J. L. Velázquez, *Point dynamics in a singular limit of the Keller–Segel model* 2: *Formation of the concentration regions*, SIAM J. Appl. Math., 64 (2004), pp. 1224–1248.
[35] R. Wedlich-Soldner, S. Altschuler, L. Wu, and R. Li, *Spontaneous cell polarization through actomyosin-based delivery of the Cdc42 GTPase*, Science, 299 (2003), pp. 1231–1235.
[36] R. Wedlich-Soldner, S. C. Wai, T. Schmidt, and R. Li, *Robust cell polarity is a dynamic state established by coupling transport and GTPase signaling*, J. Cell Biol., 166 (2004), pp. 889–900.
[37] S. Whitelam, T. Bretschneider, and N. J. Burroughs, *Transformation from spots to waves in a model of actin pattern formation*, Phys. Rev. Lett., 102 (2009), 198103.